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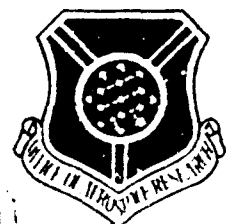
L. G. HANSCOM FIELD, BEDFORD, MASSACHUSETTS

Categorizations and Realizations of Positive Real and Biquadratic Immittance Functions

KURT H. HAASE

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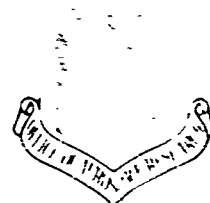
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Abstract

It is shown that a positive real immittance function $F(s)$ is of one of eight categories. The category can be recognized by the sign polarities of three test values that are functions of the coefficients of $F(s)$. If $F(s)$ is of a certain category, then $1/F(s)$ can only be of some other categories. According to the categories of $F(s)$ and $1/F(s)$ the immittance function can be realized (1) either by an RC or an RL network with positive elements, (2) by an RLC network with exclusively positive elements and an equivalent model circuit, or (3) only by model circuits. A model circuit is an RLC ladder structure with one negative branch element. The RC, RL, RLC, and model circuits have several equivalences.

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Categorizations and Realizations of Positive Real and Biquadratic Immittance Functions

1. INTRODUCTION

The problem of realizing electrical networks that have a prescribed positive real and biquadratic function as driving-point impedance or admittance has been and is still being widely discussed in the literature. The foremost reason for so many papers on the same subject lies in the fact that the problem of realization in general does not have a unique solution. Although the order of a biquadratic function is a relatively low one, we will see in this paper that there are already a great variety of circuit configurations which can realize such a function; there are still more circuits, but we will restrict ourselves to the realizations of the canonical ladder type. We feel that non-canonical realizations and realizations of the lattice and other configurations deserve a separate discussion that is intended for a later publication.

Canonical realizations of the ladder type incorporate resistive (R), inductive (L), and capacitive (C) circuit elements in a minimum number. The elements are not necessarily all positive. We will see that there are 48 ladder configurations, each of which realizes a positive real and biquadratic driving-point function. There are some groups of realizations that realize the same function; the circuits of such a group are referred to as equivalent. Those ladder realizations

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that have exclusively positive elements are generally known. As a novel contribution this paper presents a simple test method by which one can quickly find out whether or not such a ladder realization, with only positive elements, can be obtained from a biquadratic function known by its coefficients. Ladder configurations in which one element is negative are also known as "Brune Circuits." This paper shows that these circuits have several equivalences which — at least to the author's knowledge — are not yet known. The test method to be discussed also reveals whether or not such a realization can be obtained for a given function. In order to apply the test method, we first divide the positive real and biquadratic functions into eight categories. Each category — as we will show — can be realized by certain groups of equivalent circuits. Thus, when the test reveals the category to which a certain function belongs, one can immediately indicate the qualitative circuit structures without the necessity of developing the entire circuit. This is certainly an advantage in the realization problem.

Those circuits that incorporate one negative circuit element (there is only one negative element on the biquadratic level) cannot be realized, of course, from the set of only positive elements. It is, however, always possible to transform these structures further into a non-canonical configuration with only positive elements. These further transformations are reserved for a future publication. Thus the paper will present only those canonical structures, which, since they are not realizable by positive elements but are nevertheless fundamental for a further transformation, are referred to as "model circuits."

Finally, it is hoped that the present paper offers many ideas which later can also be applied to the problem of realizing driving-point functions of a higher order than the biquadratic one.

2. FUNDAMENTAL CONSIDERATIONS ON THE BIQUADRATIC IMMITTANCE FUNCTION

2.1 The Immittance Concept

Let a two-terminal network (also referred to as a "one-port") be generated by a sinusoidal electromotive source as shown in Figure 1. Assume that the "black box" in the figure does not incorporate other than linear and passive RLC elements. The current J excited by the voltage E will then be sinusoidal also. The driving-point behavior of the black box can be described sufficiently by the quotient between E and J with no need to consider the circuit in the box, the elements involved, their number, or their interconnections. The term "quotient" is mathematically somewhat neutral as long as we do not clearly define what is in the numerator and in the denominator. A very similar situation is true from the physical point of view. When we presume that E and J are references to norm units, with only

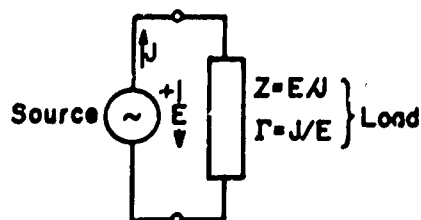


Figure 1. Voltage E and Current I at the Terminals of a One-port

the latter ones measured in volts and amperes so that E and J are normalized, then the quotient is also normalized. Hendrick Bode (1945) introduced the word "immittance" as the neutral physical interpretation of the neutral mathematical term "quotient" between E and J . The term can either be interpreted as impedance $Z(s) = E/J$ or as admittance $\Gamma(s) = J/E$. We will denote in this paper the immittance as $F(s)$. Although there is a great advantage in the use of the term immittance, the concept has been used very rarely over the years. It is not listed in many competent technical dictionaries and there are still, from the author's observation, many engineers and technicians who have never heard of it. We will base all our discussions of the nature of positive real biquadratic driving-point functions on the immittance concept.

In general the positive real immittance function $F(s)$ is a function of the complex frequency variable

$$s = \sigma + j\omega. \quad (1)$$

In Eq. (1), $\omega = 2\pi f_r$, where f_r is referred to a norm frequency f_N , measured in cps for instance, so that $f_r = f/f_N$ is a ratio with no physical dimension. The real component σ in Eq. (1) is also referred and can for many discussions be considered as a "dummy component." With $\sigma = 0$ the variable s can conveniently be used to avoid the imaginary unit i .

2.2 Brune's Statements on the Positive Realness Applied to the Immittance Function

Otto Brune (1931) was the first one to state the necessary and sufficient conditions to be imposed on an analytical function in order that this function represent the driving-point impedance of a network that incorporates only linear and passive (not necessarily positive) elements. Such a function $Z(s)$ is in Brune's definition positive real (pr). With $Z(s)$ pr, $\Gamma(s) = 1/Z(s)$ is also pr. Therefore Brune's statements can immediately be applied to the immittance function $F(s)$. In this concept they are:

An immittance function $F(s)$ is positive real if, and only if $F(s)$ is real for real s , and if for real and non-negative s $F(s)$ is also real and non-negative, that is if

$$\operatorname{Re}(s) \geq 0 \longrightarrow \operatorname{Re} F(s) \geq 0. \quad (2)$$

The condition of positive realness can also be specified in the following way: the function $F(s)$ can have poles and zeros only in the left half of the complex s -plane including the imaginary $j\omega$ -axis. If all poles and zeros are located on the imaginary $j\omega$ -axis, then the poles have to alternate with zeros in their sequential locations, they have to be simple (multiplicity 1), and they have to have positive residues; at $\omega = 0$ there must be a pole or a zero and also at $\omega = \pm \infty$. Poles and zeros can only appear as conjugate complex pairs if they are not located on the real σ -axis. When s is substituted by $\sigma + j\omega$ with $\sigma = 0$, then $\operatorname{Re} F(j\omega) \geq 0$ for all $\pm \omega$.

2.3 The Positive Real Biquadratic Immittance Function

A rational function of the intermediate variable x is referred to as biquadratic when a quadratic polynomial, say $ax^2 + bx + c$, is divided by another quadratic polynomial $dx^2 + ex + f$. Thus the function

$$\frac{ax^2 + bx + c}{dx^2 + ex + f} = \frac{a}{d} \cdot \frac{x^2 + bx/a + c/a}{x^2 + ex/d + f/d} \quad (3)$$

is a biquadratic (rational) function. Evidently, when we are only interested in the dependency on the variable x , then it is sufficient to consider only the function on the right side of Eq. (3) without the factor a/d , which is only a positive and real scale factor. This fraction we call a "normalized biquadratic rational function."

In this paper we discuss the biquadratic immittance function.

$$F(s) = \frac{s^2 + N_1s + N_0}{s^2 + D_1s + D_0}. \quad (4)$$

We use the notation $F(s)$ further only when it has the form presented in Eq. (4). This means that it is normalized and biquadratic. We presume that the coefficients N_0, N_1 (N standing for 'numerator'), and the coefficients D_0, D_1 (D standing for 'denominator') are all positive and real and that none of them degenerates to 0 or ∞ .

We also presume that $F(s)$ is a positive real function. For this, positive and real coefficients N_0, N_1, D_0, D_1 are necessary, but these characteristics of the coefficients are not sufficient to ensure that $F(s)$ is pr. Although the conditions for

positive realness of a biquadratic function are known, we would like to derive them as follows:

We postulate that

$$\operatorname{Re} F(j\omega) \geq 0 \text{ for all } \omega, \quad (5a)$$

or introducing

$$\Omega = \omega^2, \quad (6)$$

we postulate that

$$\operatorname{Re} F(j\omega) \geq 0 \text{ for all } \Omega. \quad (5b)$$

Replacing s by $j\omega$ we obtain:

$$F(j\omega) = \frac{N_0 - \omega^2 + j\omega N_1}{D_0 - \omega^2 + j\omega D_1}. \quad (7)$$

Since the denominator in the fraction of Eq. (7) is complex, we multiply numerator and denominator by the factor $D_0 - \omega^2 - j\omega D_1$ and we obtain

$$F(j\omega) = \operatorname{Re} F(j\omega) + j \operatorname{Im} F(j\omega), \quad (8)$$

where

$$\operatorname{Re} F(j\omega) = \frac{\omega^4 + \omega^2(N_1 D_1 - N_0 - D_0) + N_0 D_0}{\omega^4 + \omega^2(D_1^2 - 2D_0) + D_0^2}, \quad (8a)$$

and

$$\operatorname{Im} F(j\omega) = -j\omega \frac{N_1(\omega^2 - D_0) - D_1(\omega^2 - N_0)}{\omega^4 + \omega^2(D_1^2 - 2D_0) + D_0^2}. \quad (8b)$$

Replacing ω^2 by Ω according to Eq. (6) we get

$$\operatorname{Re} F(j\omega) = \frac{\Omega^2 + \Omega(N_1 D_1 - N_0 - D_0) + N_0 D_0}{\Omega^2 + \Omega(D_1^2 - 2D_0) + D_0^2} \quad (9a)$$

and

$$\text{Im } F(j\omega) = -j\omega \frac{(N_1 - D_1)\Omega + (N_0 D_1 - N_1 D_0)}{\Omega^2 + \Omega(D_1^2 - 2D_0) + D_0^2} . \quad (9b)$$

We are interested only in the result obtained in Eq. (9a).

By its derivation from a square the denominator in Eq. (9a) is certainly positive for all positive Ω . Thus we have only to be sure that the numerator in this equation is positive for all positive Ω . The product $N_0 D_0$ is positive by the definition of the coefficients of $F(s)$ as positive. Thus only the factor $(N_1 D_1 - N_0 - D_0)$ can disturb the positiveness of the numerator. We do not have to worry when this factor is positive. But when it is negative, then the numerator is positive only when the quadratic equation

$$\Omega^2 + \Omega(N_1 D_1 - N_0 - D_0) + N_0 D_0 = 0 , \quad (10)$$

has a pair of conjugate complex solutions. It is a well known fact in the field of algebra that in this event the left side of Eq. (10) is positive for all real Ω . Equation (10) has the discriminant

$$\Delta_\Omega = (N_1 D_1 - N_0 - D_0)^2 - 4N_0 D_0 . \quad (11)$$

The discriminant has to be non-positive. It evidently can also be written in the form

$$\begin{aligned} \Delta_\Omega &= \left[N_1 D_1 - N_0 - D_0 + 2\sqrt{N_0 D_0} \right] \left[N_1 D_1 - N_0 - D_0 - 2\sqrt{N_0 D_0} \right] \\ &= \left[N_1 D_1 - (\sqrt{N_0} - \sqrt{D_0})^2 \right] \left[N_1 D_1 - (\sqrt{N_0} + \sqrt{D_0})^2 \right] \end{aligned} \quad (12)$$

It can easily be observed in Eq. (12) that Δ_Ω can be negative only if the expression in the first pair of brackets in the second line is positive and the one in the second pair is negative. If the expression in the first pair of brackets is negative, the one in the second pair is also negative; and if the expression in the second pair is positive, the one in the first pair is also positive. Hence positive realness of the bi-quadratic function $F(s)$ postulates that

$$N_1 D_1 \geq (\sqrt{N_0} - \sqrt{D_0})^2 . \quad (13)$$

The postulation (13) with its sign of inequality also holds when the numerator in Eq. (9a) equals $(\Omega + \Omega_a)(\Omega + \Omega_b)$ with Ω_a and Ω_b positive.

Any biquadratic function for which Eq. (13) holds is positive real. There are two sub-sets of pr biquadratic functions: for the one the sign $>$, for the other the sign $=$ holds. From now on we will use the notation $F(s)$ predominantly for functions of the first sub-set. When we are sure that the sign of equality in Eq. (13) holds, we will always use the notation

$$\bar{F}(s) = \frac{s^2 + \bar{N}_1 s + \bar{N}_0}{s^2 + \bar{D}_1 s + \bar{D}_0}, \quad (14)$$

for which

$$\bar{N}_1 \bar{D}_1 = (\sqrt{\bar{N}_0} - \sqrt{\bar{D}_0})^2 = \bar{N}_0^2 + \bar{D}_0^2 - 2\sqrt{\bar{N}_0 \bar{D}_0}. \quad (14a)$$

We refer to the function $\bar{F}(s)$ in Eq. (14) as a singular pr biquadratic immittance function. When interpreted as an impedance function it is generally known in the literature as "a minimum resistance function." A singular function can easily be recognized as such by the bar notation attributed to the capital letters.

A function

$$F(s) = \frac{s^2 + N_1 s + N_0}{s^2 + D_1 s + D_0}, \quad (15)$$

for which in general

$$N_1 D_1 > (\sqrt{N_0} - \sqrt{D_0})^2 \quad (15a)$$

is referred to as a non-singular pr biquadratic immittance function from now on.

The real component of a singular function $\bar{F}(s)$ according to Eq. (9a) is

$$\operatorname{Re} \bar{F}(j\omega) = \frac{\Omega^2 - 2\Omega \sqrt{\bar{N}_0 \bar{D}_0} + \bar{N}_0 \bar{D}_0}{\Omega^2 + \Omega(\bar{D}_1^2 - 2\bar{D}_0) + \bar{D}_0^2}. \quad (16)$$

The numerator in Eq. (16) equals zero when $\Omega = \sqrt{\bar{N}_0 \bar{D}_0}$.

3. CHARACTERISTICS OF THE PR BIQUADRATIC FUNCTIONS

3.1 A Categorization of pr Biquadratic Functions

Many relations between the magnitudes of the real coefficients N_0, N_1, D_0, D_1 are feasible when we postulate by the equality (14a) or by the inequality (15a) that a function $\bar{F}(s)$ or a function $F(s)$ should be positive real. One can predict that we will be able to distinguish between certain categories such that functions that are of the same category will show roughly the same dependency on the frequency variable. We categorized the pr immittance functions $F(s)$ according to similarities of $\text{Re } F(j\omega)$ in Eq. (9a). First we separated functions $F(s)$ with $N_0 > D_0$ from functions with $N_0 < D_0$. This separation is a division of the class of biquadratic pr functions $F(s)$ into two main-categories:

$$\text{main-category (a) is defined by } N_0 > D_0, \quad (17a)$$

$$\text{main-category (b) is defined by } N_0 < D_0. \quad (17b)$$

We found that when we plot $\text{Re } F(j\omega)$ vs Ω in the range $0 \leq \Omega \leq \infty$ we can distinguish between four sub-categories in each main-category. We thus come up with a total of eight categories (a1), (a2), (a3), (a4) and (b1), (b2), (b3), (b4). We show the real components of the sub-categories of main-category (a) in Figure 2 and the real components of the sub-categories of main-category (b) in Figure 3. The figures should be taken as rough sketches only.

Since

$$\text{Re } F(j0) = F(0) = N_0/D_0, \quad (18)$$

and

$$\text{Re } F(j\infty) = F(\infty) = 1, \quad (19)$$

the curves that we sketched in both figures start with the ordinate N_0/D_0 at $\Omega = 0$ and end with the ordinate 1 at $\Omega = \infty$. In both main-categories the curves in sub-category (1) show no inflections and remain within the ordinates N_0/D_0 and 1 between the frequency limits 0 and ∞ . The smallest magnitude of $\text{Re } F(j\omega)$ in category (a1) is 1 and in category (b1) it is N_0/D_0 . In both main-categories the curves show one inflection which is a maximum N_m at Ω_m in sub-category (3). Again the smallest magnitude of $\text{Re } F(j\omega)$ is 1 in category (a3) and N_0/D_0 in category (b3). In both main-categories the curves show two inflections in sub-category (2), the one is a maximum N_m at Ω_m , the other a minimum N_m at Ω_m . Thus the smallest magnitude of $\text{Re } F(j\omega)$ is N_m in categories (a2) and (b2).

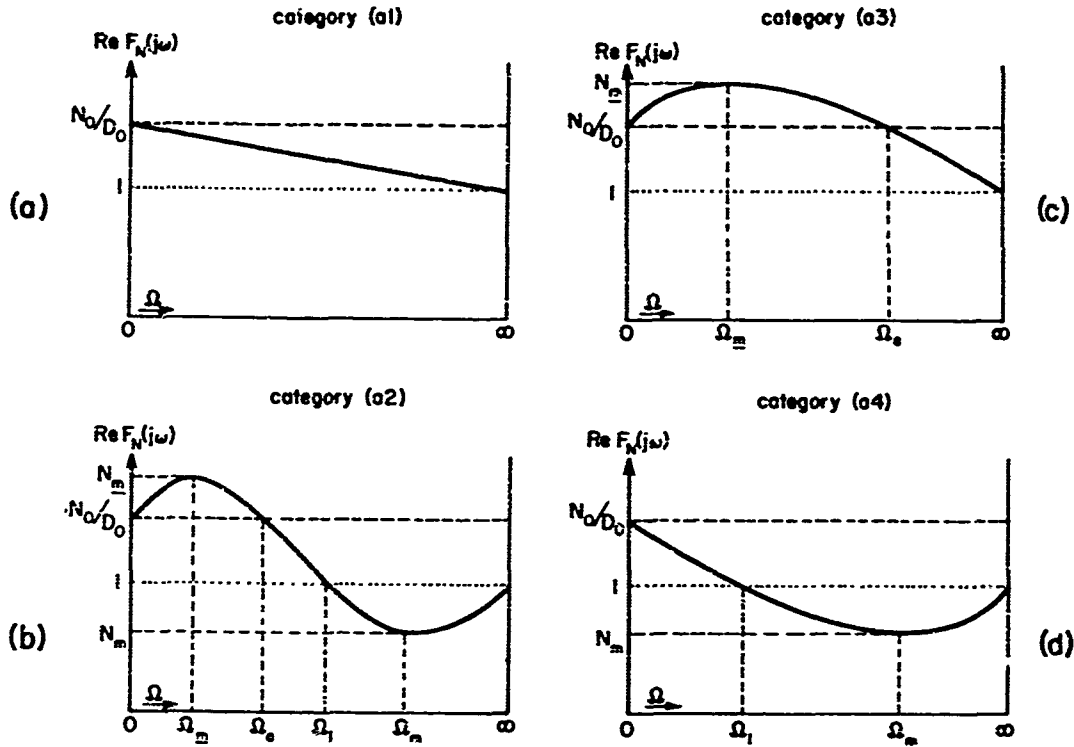


Figure 2. Sketches of $\text{Re } F(j\omega)$ vs $\Omega = \omega^2$ for $N_0/D_0 > 1$

Finally in both main-categories the curves show one inflection in sub-categories (4) which is a minimum N_m at Ω_m . Thus also in categories (a4) and (b4) the smallest magnitude of $\text{Re } F(j\omega)$ is N_m .

We also observe that in the range $0 < \Omega < \infty$ the ordinate N_0/D_0 is crossed once at Ω_a in the categories (a2), (a3), (b2), (b3); the ordinate 1 is crossed once at Ω_1 in the categories (a2), (a4), (b2), (b3). There are no crossings Ω_a and Ω_1 in sub-categories (1), and there is no crossing Ω_a in category (a4) and no crossing Ω_1 in category (b4).

There are now two techniques by which we can find out to which category a certain function belongs: either we differentiate and identify the differential quotient with 0,

$$\frac{d \text{Re } F(j\omega)}{d\Omega} \cdot \frac{d\Omega}{d\omega} = 0, \quad (20)$$

or we search for the ordinate crossings Ω_a and Ω_1 by solving the equations

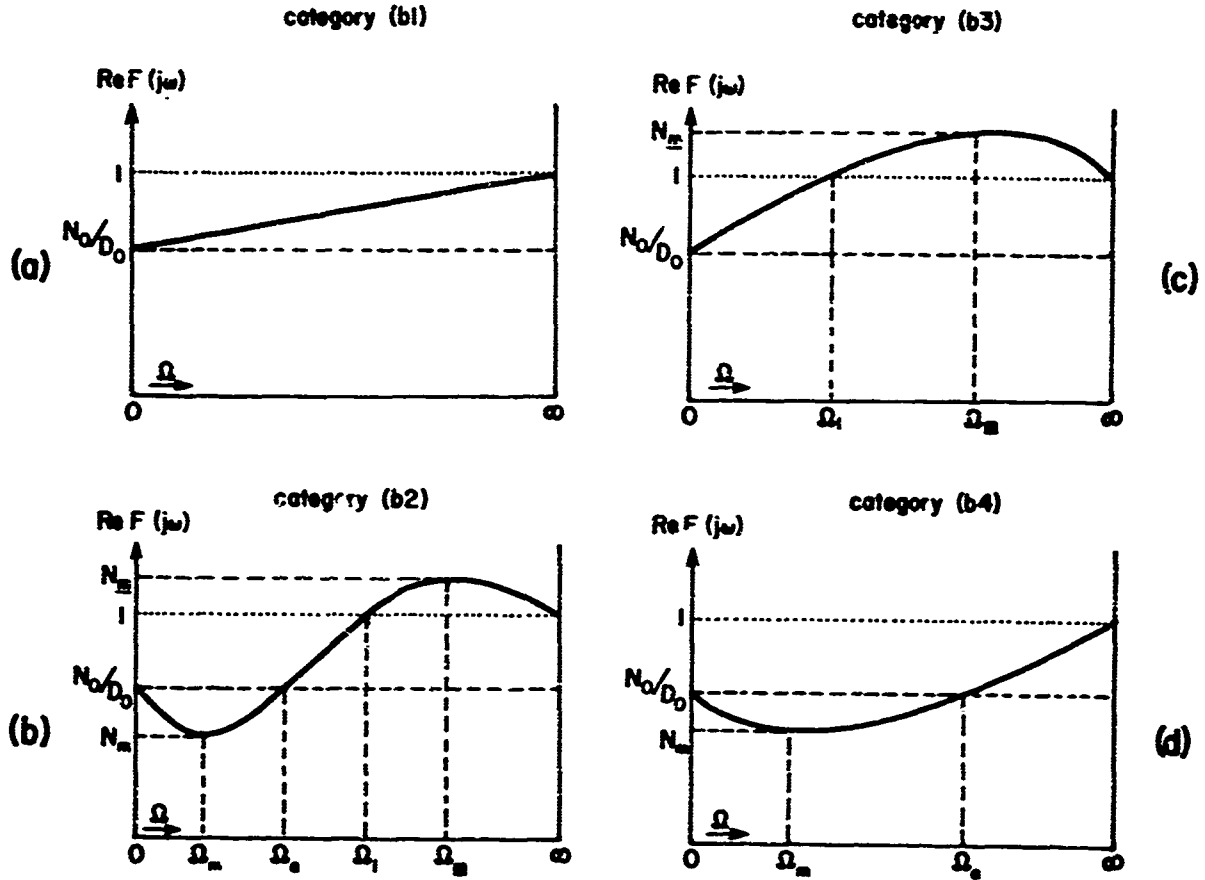


Figure 3. Sketches of $\text{Re } F(j\omega)$ vs $\Omega = \omega^2$ for $N_0/D_0 < 1$

$$\text{Re } F(j\omega) = N_0/D_0 \quad (21a)$$

and

$$\text{Re } F(j\omega) = 1. \quad (22a)$$

Quadratic Eq. (20) yields the solutions Ω_m and $\Omega_{\bar{m}}$ which, according to well known rules of algebra, have to be identified as the minimum and/or maximum frequencies. Equations (21a) and (21b), however, are linear and have the solutions

$$\Omega_a = \frac{D_0(N_0 - D_0) - D_1(N_0 D_1 - N_1 D_0)}{N_0 - D_0} \quad (21b)$$

and

$$\Omega_1 = \frac{D_0 (N_0 - D_0)}{(N_0 - D_0) - D_1 (N_1 - D_1)} . \quad (22b)$$

When there is a crossing of the ordinates N_0/D_0 and/or 1 in the range $0 < \Omega < \infty$ the solution in Eqs. (21b) and/or (22b) must be positive. Since our aim at the moment is to find a "test method" by which we can identify the category of $F(s)$, the test has to be simple and for this reason we decided to choose the second way.

First according to the definitions of the main-categories in (17a, b) we introduce the test value

$$T_0 = N_0 - D_0 . \quad (23)$$

If T_0 is positive, we have main-category 'a); if it is negative, we have main-category (b). Substituting T_0 in Eqs. (21b) and (22b) we obtain two other test values

$$T_a = D_0 T_0 - D_1 (N_0 D_1 - N_1 D_0) , \quad (24)$$

and

$$T_1 = T_0 - D_1 (N_1 - D_1) . \quad (25)$$

Since $\Omega_a = T_a/T_0$ and $\Omega_1 = D_0 T_0/T_1$, the test values T_a and T_1 simply by their sign polarity identify the respective sub-category of the function $F(s)$. This is shown in Table 1.

Table 1. Sign Polarities of \bar{u} - Test Values T_0 , T_a , and T_1 in the Eight Categories

Test Value	Category							
	(a1)	(a2)	(a3)	(a4)	(b1)	(b2)	(b3)	(b4)
T_0	+	+	+	+	-	-	-	-
T_a	-	+	+	-	+	-	+	-
T_1	-	+	-	+	+	-	-	+

Each sign combination appears only once in any column of Table 1, and since we have 3 test values, each of them either + or -, there are $2^3 = 8$ variations possible corresponding to eight categories.

In Table 1 and in further discussions in the text we prefer to use the notation + for ≥ 0 and - for ≤ 0 . The value 0 can always be counted as + or as -. For instance if $N_0 = D_0$ and thus $T_0 = 0$, the function can be considered as being of main-category (a) or of main-category (b).

3.2 Minimum N_m and Minimum Frequency Ω_m in Sub-Categories (2) and (4)

By Eq. (20) and using the test values T_0 , T_a , T_1 defined in Eqs. (23), (24), and (25) we obtain the frequency Ω_m at which the minimum N_m in the sketches in Figures 2 and 3 appears as

$$\Omega_m = (T_0 \div c_2) D_0 / T_1, \quad (26)$$

with

$$c_2 = \sqrt{T_0^2 - T_a T_1 / D_0}, \quad (26a)$$

when $F(s)$ is of sub-category (2), that is, if it is of category (a2) or (b2).

If $F(s)$ is of sub-category (4), that is, if it is of either category (a4) or (b4), then,

$$\Omega_m = (T_0 \div c_4) D_0 / T_1, \quad (27)$$

with

$$c_4 = \sqrt{T_0^2 \div |T_a T_1| / D_0}. \quad (27a)$$

The constants c_2 and c_4 in Eqs. (26a) and (27a) are positive. Test values T_0 and T_1 are both positive in Eq. (26) if $F(s)$ is of category (a2) and both negative if it is of category (b2). In Eq. (27) T_1 is always positive, but T_0 is positive if $F(s)$ is of category (a4) and negative if it is of category (b4).

The minimum frequency Ω_m obtained in Eqs. (26) or (27) must always be positive.

The minimum N_m is obtained by substituting Ω_m into Eq. (9a).

Thus

$$N_m = \frac{(\Omega_m^2 \div N_0 D_0) \div \Omega_m (N_1 D_1 - N_0 - D_0)}{(\Omega_m^2 + D_0^2) \div \Omega_m (D_1^2 - 2D_0)} \quad (28)$$

3.3 Some Other Characteristics of the Biquadratic Functions

We are now able to recognize the category to which a function $F(s)$, known by the numerals in its coefficients, belongs. There are some characteristics that are of further interest. Assume, for instance, that we keep the coefficients N_0 and D_0 constant and we ask: what are the ranges for the magnitudes of the coefficients N_1 and D_1 so that the function remains within its original category?

In the denominator of $F(s)$ let us replace

$$D_1 = 2r\sqrt{D_0} \quad (29)$$

with r as a positive and real constant. It is always possible to refer D_1 in this way to D_0 . Then by Eq. (24)

$$T_a/D_0 = N_1 D_1 - (4r^2 N_0 - T_0) \quad (30)$$

and by Eq. (25)

$$T_1 = (D_1^2 \div T_0) - N_1 D_1 \quad (31)$$

When we postulate that T_a is positive, then

$$\frac{4r^2 N_0 - T_0}{D_1} \leq N_1 < \infty, \quad (32a)$$

and when we postulate that T_a is negative, then

$$0 < N_1 \leq \frac{4r^2 N_0 - T_0}{D_1} \quad (32b)$$

When we postulate that T_1 is positive, then

$$0 < N_1 \leq \frac{4r^2 D_0 \div T_0}{D_1} \quad (33a)$$

and when we postulate that T_1 is negative, then

$$\frac{4r^2 D_0 + T_0}{D_1} \leq N_1 < \infty. \quad (33b)$$

Note that for

$$r^2 = 0.5 \quad (34a)$$

$$4r^2 N_0 - T_0 = 4r^2 D_0 + T_0 = N_0 + D_0. \quad (34b)$$

In sub-categories (1) and (2) T_a and T_1 are of the same sign polarities, both positive or negative. It can easily be shown that in main-category (a) where T_0 is positive the results of the pair of Eqs. (32a) and (33a) are contradictory when $r^2 > 0.5$, and in main-category (b) where T_0 is negative the results of the pair of these equations are contradictory when $r^2 < 0.5$. Similarly the results of the pair of Eqs. (32b) and (33b) are contradictory in main-category (a) when $r^2 < 0.5$ and in main-category (b) when $r^2 > 0.5$. Hence

$$\text{in categories (a1) and (b1) only } r^2 \geq 0.5 \text{ is acceptable,} \quad (35a)$$

$$\text{in categories (a2) and (b2) only } r^2 \leq 0.5 \text{ is acceptable.} \quad (35b)$$

$$\text{For the other categories any magnitude of } r^2 \text{ is acceptable.} \quad (35c)$$

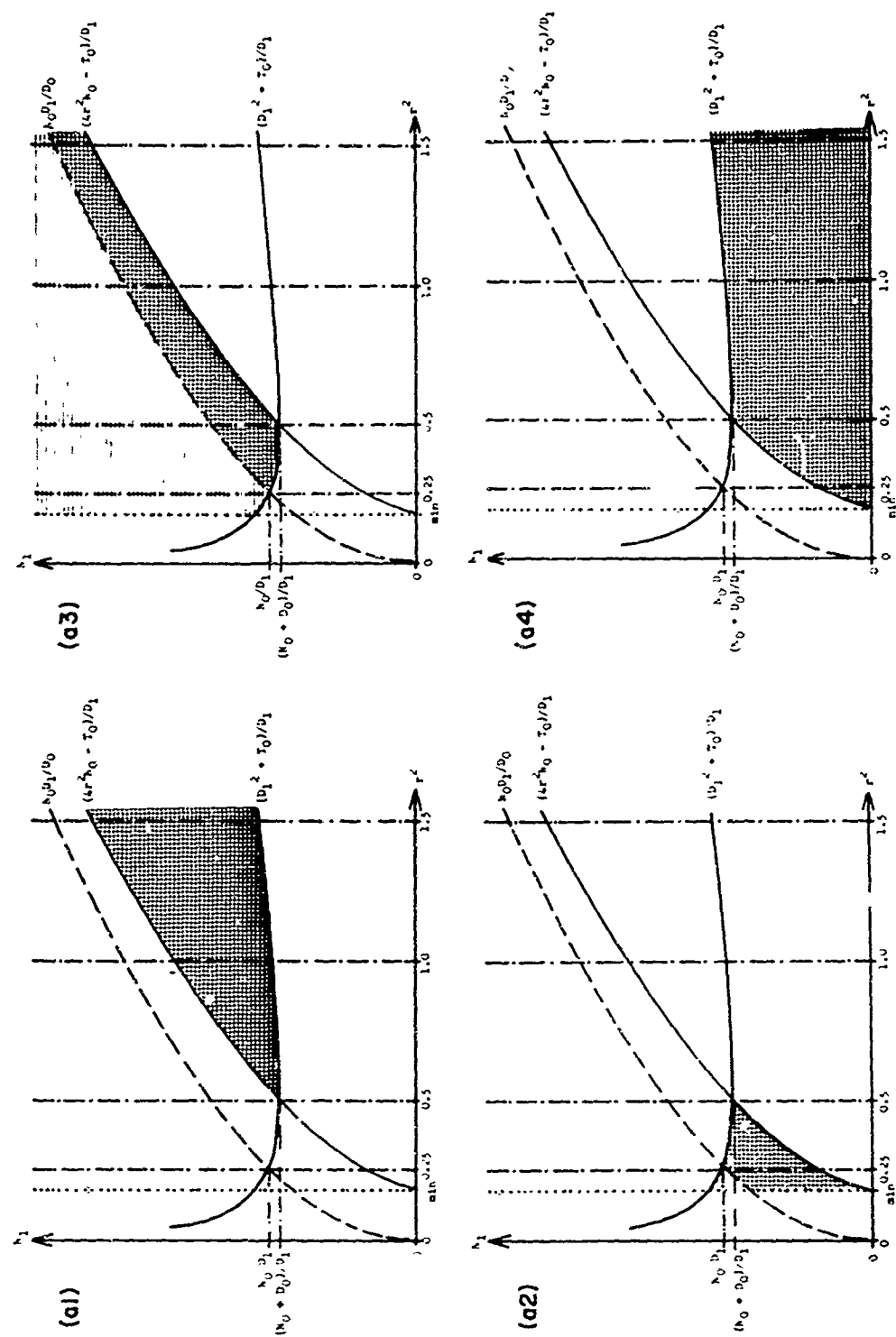
With D_1 referred to D_0 by Eq. (29) we show the ranges of N_1 in Table 2. It is interesting to observe that the discriminant $4D_0(r^2-1)$ of the denominator of $F(s)$ is not important in our categorization; thus we do not pay attention to whether r^2 is greater or smaller than 1. We rather observe whether r^2 is greater or smaller than 0.5.

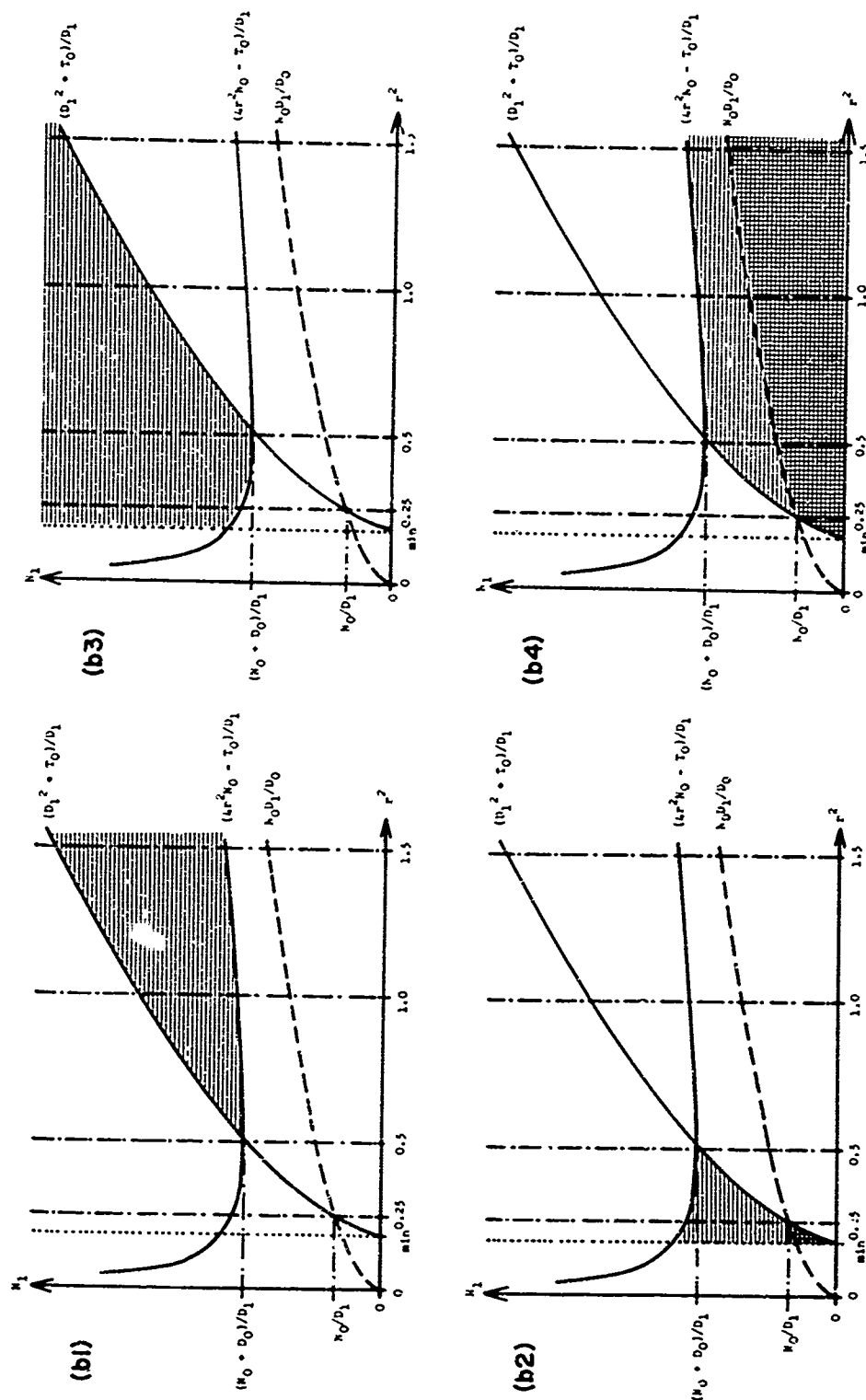
3.4 The Sign Polarities of $(N_0 D_1 - N_1 D_0)$ and $(N_1 - D_1)$

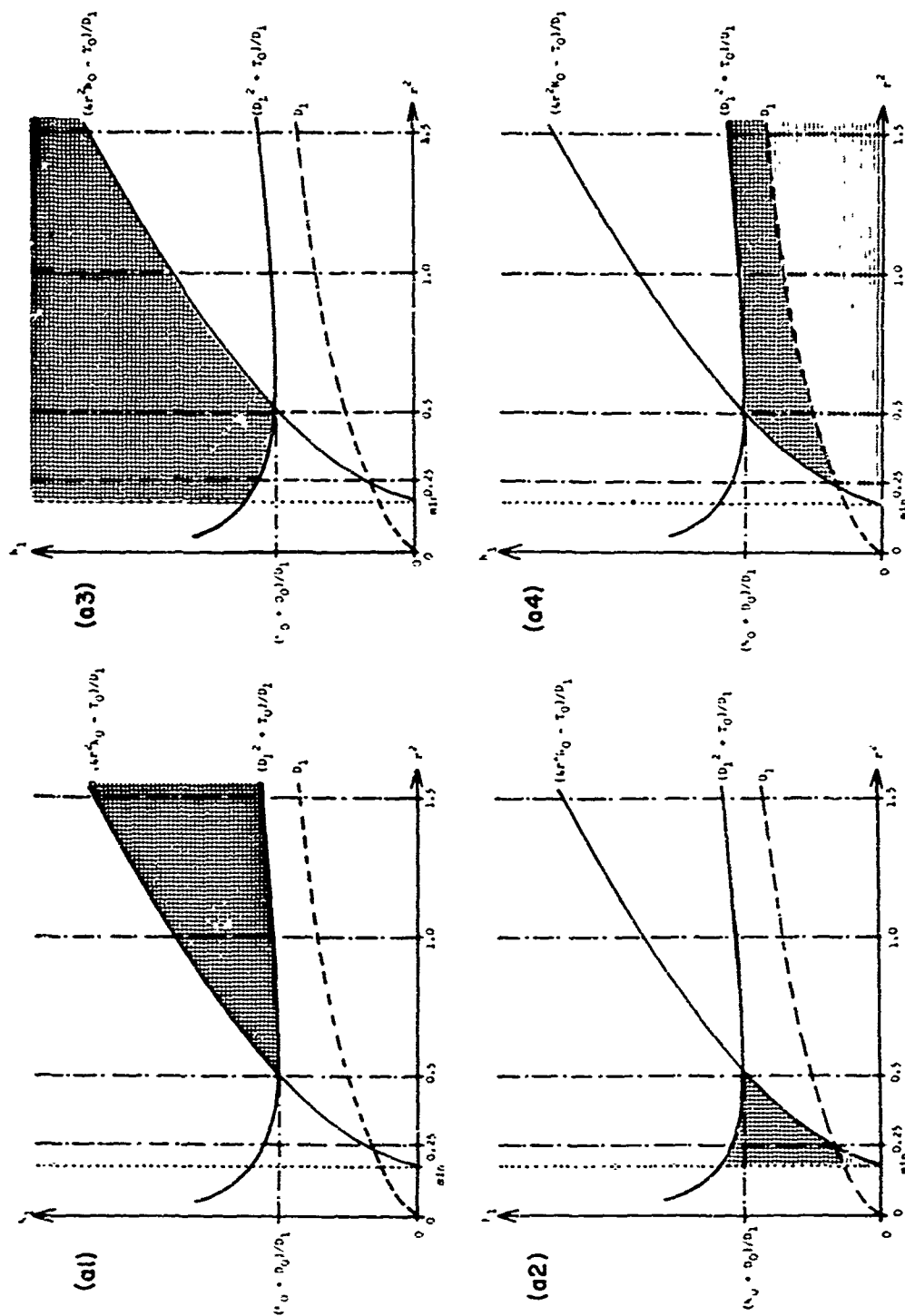
For later discussions we are also interested in the sign polarities of the expressions $(N_0 D_1 - N_1 D_0)$ and $(N_1 - D_1)$ which are implied, for instance, in the formulas for the test values T_a and T_1 . Assuming a fixed pair of coefficients N_0 and D_0 we show the sign polarities of $(N_0 D_1 - N_1 D_0)$ in Figure 4a for main-category (a) and in Figure 4b for main-category (b). The sign polarities of $(N_1 - D_1)$ are shown in Figure 5a for main-category (a) and in Figure 5b for main-category (b). In these figures we have plotted the coefficient N_1 vs the factor r^2 . Cross shading indicates that the difference under investigation is positive, horizontal shading indicates that the difference is negative.

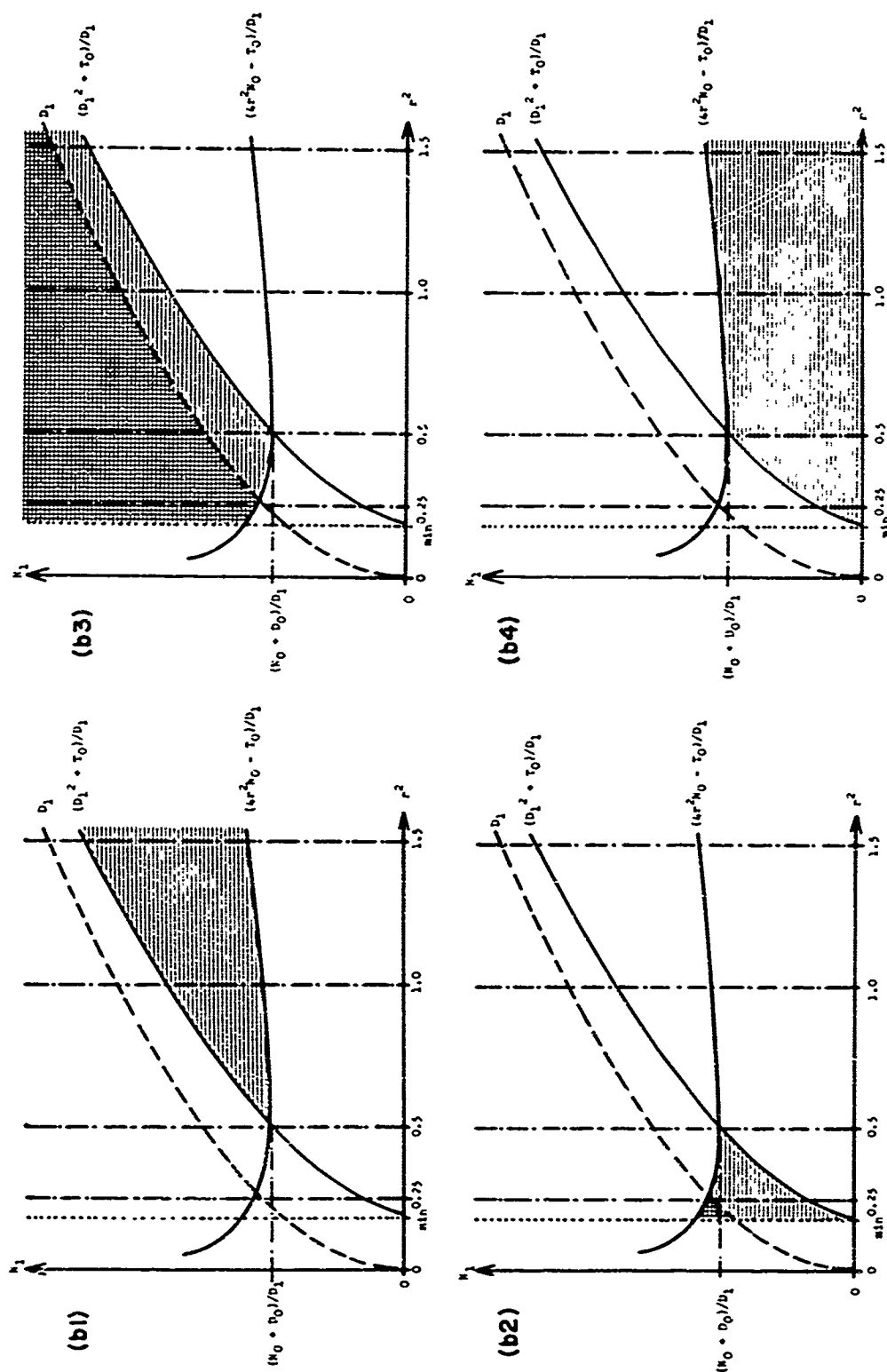
Table 3. Allowed Ranges for the Coefficient N_1 Depending on the Category and on the Constant $r^2 = 4D_0/D_1^2$

Category	$r^2 > 0.5$			$r^2 < 0.5$		
	$(D_1^2 + T_0)/D_1$	$\leq N_1 \leq$ None $\geq N_1$ $\geq N_1$	$(4r^2 N_0 - T_0)/D_1$	$(4r^2 N_0 - T_0)/D_1$	None $\leq N_1 \leq$ $\geq N_1$ $\geq N_1$	$(D_1^2 + T_0)/D_1$
(a1)						
(a2)						
(a3)						
(a4)						
(b1)						
(b2)						
(b3)						
(b4)						

Figure 4a. Sign Polarity of $N_0 D_1 N_1 D_0$ in Categories (a1), ..., (a4)

Figure 4b. Sign Polarity of $N_0D_1 - N_1D_0$ in Categories (b1), ..., (b4)

Figure 5a. Sign Polarity of $N_1 - D_1$ in Categories (a1), ..., (a4)

Figure 5b. Sign Polarity of N_1-D_1 in Categories (b1), ..., (b4)

We find that

$(N_0D_1 - N_1D_0)$ is positive only in categories (a1) and (a4), and
is negative only in categories (b1) and (b3).

$(N_1 - D_1)$ is positive only in categories (a1) and (a3), and
is negative only in categories (b1) and (b4).

In all other categories the differences can be positive or negative. Different sign areas are separated by the dashed lines corresponding to $(N_0D_1 - N_1D_0) = 0$ and $(N_1 - D_1) = 0$ respectively.

3.5 The Category of the Inverse Function $F'(s) = 1/F(s)$

Assume we have identified the category of a particular function $F(s)$ by the sign polarities of the test values T_0 , T_a , and T_1 as defined in Eqs. (23), (24), and (25). It is trivial of course to identify the inverse function

$$F'(s) = 1/F(s) = \frac{s^2 + N'_1s + N'_0}{s^2 + D'_1s + D'_0} = \frac{s^2 + D_1s + D_0}{s^2 + N_1s + N_0} \quad (35)$$

in absolutely the same way. Table 1 can also be applied to the sign polarities of

$$T'_0 = N'_0 - D'_0,$$

$$T'_a = D'_0T'_0 - D'_1(N'_0D'_1 - N'_1D'_0),$$

$$T'_1 = T'_0 - D'_1(N'_1 - D'_1).$$

But instead of determining the categories of the inverse functions individually, we would rather like to know: is there a unique relation between the category of $F(s)$ and that of $F'(s)$? Can we say: when $F(s)$ is of this category then $F'(s)$ is of that one? As a matter of fact, the answer to this question is not trivial.

It is trivial that Eq. (35) reveals

$$N'_0 = D_0, \quad (36a)$$

$$N'_1 = D_1, \quad (36b)$$

$$D'_0 = N_0, \quad (36c)$$

$$D'_1 = N_1. \quad (36d)$$

It is also trivial that

$$T'_0 = -T_0, \quad (37)$$

that is, if $F(s)$ is of main-category (a), then $F'(s)$ is of main-category (b), and vice versa. It is also trivial that

$$(N'_0 D'_1 - N'_1 D'_0) = -(N_0 D_1 - N_1 D_0), \quad (38)$$

and that

$$(N'_1 - D'_1) = -(N_1 - D_1). \quad (39)$$

Since, however, by these last two relations

$$T'_a = N_1 (N_0 D_1 - N_1 D_0) - N_0 T_0, \quad (40)$$

and

$$T'_1 = N_1 (N_1 - D_1) - T_0, \quad (41)$$

the relations between T_a and T'_a and between T_1 and T'_1 are far less obvious.

By Eqs. (24) and (40) we can relate

$$T'_a = -[N_1 T_a + (N_0 D_1 - N_1 D_0) T_0] / D_1, \quad (42a)$$

or

$$T_a = -[D_1 T'_a + (N_0 D_1 - N_1 D_0) T_0] / N_1, \quad (42b)$$

and

$$T'_a = -[T_a N_0 + (N_0 D_1 - N_1 D_0)^2] / D_0, \quad (42c)$$

or

$$T_a = -[T'_a D_0 + (N_0 D_1 - N_1 D_0)^2] / N_0. \quad (42d)$$

By Eqs. (25) and (41) we can relate

$$T_1' = -[N_1 T_1 - (N_1 - D_1) T_0] / D_1, \quad (43a)$$

or

$$T_1 = -[D_1 T_1' - (N_1 - D_1) T_0] / N_1, \quad (43b)$$

and

$$T_1' = (N_1 - D_1)^2 - T_1, \quad (43c)$$

or

$$T_1 = (N_1 - D_1)^2 - T_1'. \quad (43d)$$

It follows from Eq. (42c) that T_a' is certainly negative if T_a is positive. It also follows from Eq. (43c) that T_1' is positive if T_1 is negative. By Table 1 we recognize that we can make

Statement 1:

If $F(s)$ is of category (a3), then $F'(s)$ is of category (b4), and if $F(s)$ is of category (b3), then $F'(s)$ is of category (a4).

We are able to show that if $F(s)$ is of category (a1), then $F'(s)$ cannot be of either category (b2) or (b3). According to Table 1, T_0 is positive and T_1 negative in this case. Figure 5a shows that $(N_1 - D_1)$ is positive. Hence, by Eq. (43a) the test value T_1' of $F'(s)$ is positive. This, however, contradicts the information given by Table 1 that in categories (b2) and (b3) the test value T_1' has to be negative.

On the other hand, if $F(s)$ is of category (a1), then $F'(s)$ can be either of category (b1) or (b4). In these categories T_1' is negative according to Table 1 as well as according to what we found in the preceding paragraph. The test value T_a' is positive when $F'(s)$ is of category (b1). When $F(s)$ is of category (a1) then the test value T_a is negative. In this category, according to Figure 4a and by Table 1, $(N_0 D_1 - N_1 D_0)$ and T_0 are both positive. Hence, by Eq. (42b), T_a' is positive as it should be. If $F'(s)$ is of category (b4) and $F(s)$ of category (a1), then T_a' has to be negative. Since Figure 4b allows both polarities for $(N_0 D_1 - N_1 D_0)$ in category (b4), T_a negative can be obtained according to Eq. (42b). Thus we can make

Statement 2:

If $F(s)$ is of category (a1), then $F'(s)$ can be either of category (b1) or (b4).

If $F(s)$ is of category (b1), then $F'(s)$ can be either of category (a1) or (a4).

The second part of Statement 2 can be proved in a way similar to the proof of the first part.

We are able to show that if $F(s)$ is of category (a2), then $F'(s)$ cannot be of either category (b1) or (b3). In this case T'_a is positive. Then, according to Eq. (42d), T_a is negative. But Table 1 postulates that T_a is positive in category (a2). This is a contradiction.

On the other hand, if $F(s)$ is of category (a2), then $F'(s)$ can be either of category (b2) or (b4). For both categories T'_a must be negative according to Table 1. Figure 4b allows both polarities for the difference $(N_0D_1 - N_1D_0)$ in categories (b2) and (b4). According to Eq. (42b), T_a can be positive as it should be. Figure 5b allows both polarities also for $(N_1 - D_1)$. In category (a2) T_1 is positive. According to Eq. (43a) T'_1 can be negative and according to Eq. (43b) T_1 can be positive. We can make

Statement 3:

If $F(s)$ is of category (a2), then $F'(s)$ can be either of category (b2) or (b4).
If $F(s)$ is of category (b2), then $F'(s)$ can be either of category (a2) or (a4).

Again the second part of Statement 3 can be proved similarly to the proof of the first part.

We can show that if $F(s)$ is of category (a4), then $F'(s)$ can be of category (b4) and vice versa. According to Figure 4a the difference $(N_0D_1 - N_1D_0)$ is positive for $F(s)$ of category (a4). When we eliminate this difference in Eqs. (42a) or (42b), we find that it has the opposite polarity of T_a and T'_a which are both negative for categories (a4) and (b4). We get the same result by Eqs. (42c) or (42d). In categories (a4) and (b4) T_1 and T'_1 are both positive. Hence $T_1 + T'_1$ is positive. This is verified by Eqs. (43c) and (43d). Thus if $F(s)$ is of category (a4), then $F'(s)$ can be of category (b4) and vice versa. Statements 1, 2, and 3 prove, however, that if $F(s)$ is of category (a4), then $F'(s)$ can also be of one of the categories (b1), (b2), or (b3); and when $F(s)$ is of category (b4) then $F'(s)$ can be of any of the categories (a1), (a2), or (a3). Hence we can make

Statement 4:

When $F(s)$ is of category (a4), then $F'(s)$ can be of any of the four sub-categories in main-category (b). When $F(s)$ is of category (b4), then $F'(s)$ can be of any of the four sub-categories of main-category (a).

We summarize the four statements in Tables 3a and 3b.

Table 3a. Categories of $F'(s)$ for Certain Categories of $F(s)$

If $F(s)$ is of category	then $F'(s) = 1/F(s)$ is of either one of the categories
(a1)	(b1), (b4)
(a2)	(b2), (b4)
(a3)	(b4)
(a4)	(b1), (b2), (b3), (b4)

Table 3b. Categories of $F(s)$ for Certain Categories of $F'(s)$

If $F(s)$ is of category	then $F'(s) = 1/F(s)$ is of either one of the categories
(b1)	(a1), (a4)
(b2)	(a2), (a4)
(b3)	(a4)
(b4)	(a1), (a2), (a3), (a4)

4. THE DECOMPOSITION OF POSITIVE REAL IMMITTANCE FUNCTIONS

4.1 Terminology and Purpose of a Decomposition

When we present a pr immittance function as a sum such that each member of the sum is again a pr immittance function, then we refer to the sum as a decomposition. A sum of pr immittance functions is always a pr function; but conversely, the decomposition of a pr function does not necessarily have members which are all pr. Thus, in decomposing a pr function, we always have to make sure that the members of the decomposition are all pr.

The purpose of decomposing a pr immittance function is to obtain a sum in which the members are in some way simpler than in the un-decomposed function. In this paper we decompose the biquadratic functions first into two members; the first one is a positive constant. A constant can be considered as a degenerated pr immittance function of the most primitive form. It is pr since it is defined as positive. Thus, the first member in the decomposition certainly is simpler than the biquadratic function since it does not depend on the variable s . The second member is a pr frequency function in the following decompositions.

4.2 The Decomposition of the Biquadratic Impedance Function $F(s)$

Let us return for a moment to the sketches in Figures 2 and 3. Obviously, in all these presentations of $\text{Re } F(j\omega)$ we are able to shift the abscissa axis upward and in parallel up to a certain limit, without hurting the postulation that $\text{Re } F(j\omega) \geq 0$ for all positive Ω . This limit in categories (a1) and (a3) is 1, in categories (b1) and (b3) it is N_0/D_0 , and in categories (a2), (a4), (b2), (b4) it is N_m . Thus

if $F(s)$ is of category (a1) or (a3)

$$F(s) = 1 + F_{ak}(s), \quad (44a)$$

if $F(s)$ is of category (b1) or (b3)

$$F(s) = N_0/D_0 + F_{bk}(s), \quad (44b)$$

if $F(s)$ is of category (a2), (a4), (b2), or (b4)

$$F(s) = N_m + (1 - N_m)\bar{F}(s). \quad (44c)$$

When in parts (a) and (c) in Figure 2 we shift the abscissa up to 1 on the ordinate, then the real component becomes zero at $\Omega = \infty$. Hence according to Eq. (44a)

$$\text{Re } F_{ak}(j\omega) = 0 \text{ for } \Omega = \infty.$$

But since

$$F(\infty) = 1,$$

then

$$F_{ak}(\infty) = 0 \text{ and } \text{Im } F_{ak}(j\infty) = 0 \text{ also.}$$

Hence $F_{ak}(s)$ is no longer a biquadratic function.

When in parts (a) and (c) in Figure 3 we shift the abscissa up to the ordinate N_0/D_0 , then the real component becomes zero at $\Omega = 0$. Hence according to Eq. (44b)

$$\text{Re } F_{bk}(j\omega) = 0 \text{ for } \Omega = 0.$$

But since

$$F(0) = N_0/D_0,$$

then

$$F_{bk}(0) = 0 \text{ and } \text{Im } F_{bk}(j0) = 0, \text{ also.}$$

Hence $F_{bk}(s)$ is also no longer a biquadratic function.

When in parts (b) and (d) in Figures 2 and 3 we shift the abscissa up to the ordinate N_m , then the real component becomes a double zero at Ω_m . Since neither $\bar{F}(0)$ nor $\bar{F}(\infty)$ degenerates to 0 or ∞ , they will both be finite. Thus the component $(1 - N_m)\bar{F}(s)$ in Eq. (44c) will still be a biquadratic function. But, as we saw at the end of Section 1.3, $\bar{F}(s)$ will be singular.

It is the aim of the next three sections to investigate the functional components $F_{ak}(s)$, $F_{bk}(s)$, and $(1 - N_m)\bar{F}(s)$ in more detail.

4.2.1 THE FUNCTIONAL COMPONENT $F_{ak}(s)$ IN A DECOMPOSITION OF A FUNCTION $F(s)$ OF CATEGORIES (a1) OR (a3)

By Eq. (44a) we obtain

$$F_{ak}(s) = (N_1 - D_1) \frac{s + (N_0 - D_0)/(N_1 - D_1)}{s^2 + D_1 s + D_0}. \quad (45)$$

Evidently $F_{ak}(s)$ is no longer biquadratic. Let us make sure that it is pr. A necessary condition for this is that all of its coefficients must be positive.

The coefficients D_0 and D_1 are positive by their definitions. The difference $N_0 - D_0$ is positive since $F(s)$ is of the main category (a). The difference $N_1 - D_1$ is positive in categories (a1) and (a3) according to Figure 5a. Thus the numerator coefficient $(N_0 - D_0)/(N_1 - D_1)$ in Eq. (45) and also the factor $(N_1 - D_1)$ in front of the fraction are positive. It is easy to find that

$$\text{Re } F_{ak}(s) = \frac{D_0 T_0 - \Omega T_1}{\Omega^2 + \Omega(D_1^2 - 2D_0) + D_0^2}. \quad (46)$$

The denominator in Eq. (46) is the same as in Eq. (9a), hence it is positive for all positive Ω . Table 1 shows that in categories (a1) and (a3) the test value T_1 that appears in the numerator of Eq. (46) is negative. Hence the numerator in Eq. (46) is also positive for all positive Ω , and thus the component $F_{ak}(s)$ is pr.

Consulting Table 2 we find that for categories (a1) and (a3) the square constant r^2 is allowed to be greater than 0.5. Thus it may also be greater than 1, in which event the discriminant of the denominator

$$\Delta_D = D_1^2 - 4D_0 = (r^2 - 1) 4D_0 \quad (47)$$

becomes positive. Then introducing

$$a_0^2 = \sqrt{D_0} (r - \sqrt{r^2 - 1}), \quad (48a)$$

$$a_1^2 = \sqrt{D_0} (r + \sqrt{r^2 - 1}), \quad (48b)$$

$$b_1^2 = T_0 / (N_1 - D_1), \quad (48c)$$

we have

$$F_{ak}(s) = (N_1 - D_1) \frac{s + b_1^2}{(s + a_0^2)(s + a_1^2)}. \quad (48)$$

Note that this situation is a possible, but not a necessary one. It may also happen that between the positive constants a_0^2 , a_1^2 , and b_1^2 the condition

$$a_0^2 < b_1^2 < a_1^2 \quad (48d)$$

holds. But note that this additional condition (48d) is not a necessity always coexistent with a positive discriminant Δ_D and with Eqs. (48a, b, c). But if all Eqs. (48a, ..., d) hold, then $F_{ak}(s)$ as represented in Eq. (48) is a function with two poles $-a_0^2$ and $-a_1^2$ located on the real σ -axis with a zero $-b_1^2$ between them. Thus, with poles and zeros alternating in their location on the σ -axis, $F_{ak}(s)$ in this case is a particular pr function. It is one of the class of pr impedance and/or admittance functions that can be realized by either an RC or an RL network. These functions have been discussed in an earlier scientific paper of the author [Haase (1963a)]. Since in the present one we shall very often refer to this other paper, its respective tables have been reprinted and included as Tables A-1, ..., A-11 in the appendix.

The function in Eq. (48), for which all Eqs. (48a, ..., d) must hold, is a function of the type

$$Q_4^{-1} = \frac{s + b_1^2}{k(s + a_0^2)(s + a_1^2)} \quad (\text{see Table A-2 in the appendix}),$$

for which

$$k = 1 / (N_1 - D_1). \quad (48e)$$

If Eq. (48d) does not hold for the function $F_{ak}(s)$ as it is presented in Eq. (48), or if $r^2 < 1$, so that Eq. (48) does not exist, then we propose to decompose the inverse function $1/F_{ak}(s)$ as follows:

$$\begin{aligned} 1/F_{ak}(s) &= \frac{s^2 + D_1 s + D_0}{(N_1 - D_1) [s + T_0 / (N_1 - D_1)]} \\ &= k_1 s + k_2 \frac{s + a_1^2}{s + b_1^2}. \end{aligned} \quad (49)$$

By the second right side of Eq. (49)

$$1/F_{ak}(s) = k_1 \frac{s^2 + (k_2/k_1 + b_1^2)s + k_2 a_1^2/k_1}{s + b_1^2}.$$

Comparing the coefficients in the numerator and the denominator and the factor in front of the fraction with the first right side in Eq. (49) we obtain:

$$k_1 = 1/(N_1 - D_1). \quad (49a)$$

$$k_2 = \frac{D_1(N_1 - D_1) - T_0}{(N_1 - D_1)^2} = \frac{-T_1}{(N_1 - D_1)^2} \quad (49b)$$

$$a_1^2 = \frac{D_0(N_1 - D_1)}{-T_1}, \quad (49c)$$

$$b_1^2 = \frac{T_0}{N_1 - D_1}. \quad (49d)$$

The function $F_{ak}(s)$ is presently supposed to be of category (a1) or (a3). Thus T_0 , $-T_1$, and $N_1 - D_1$ are all positive. Hence all right side in Eqs. (49a, ..., d) are positive. Both components in the second right side of Eq. (49) are pr functions. The first component $k_1 s$ is of the type

$$Q_1 = k_1 s \quad (\text{see Table A-3 in the appendix}),$$

where k_1 is given in Eq. (49a).

Provided that

$$a_1^2 > b_1^2, \text{ or } a_1^2 - b_1^2 \text{ is positive,} \quad (49e)$$

the second component is a function of the type

$$P_3 = k_2 \frac{s + a_1^2}{s + b_1^2} \quad (\text{see Table A-2 in the appendix}).$$

We proved that the second component is pr. But we now must prove that it is of the type P_3 with its zero a_1^2 left of its pole b_1^2 on the negative σ -axis. This has to be the case when the discriminant Δ_D given in Eq. (47) is negative, and/or when the discriminant Δ_D is positive, but the zero b_1^2 is not located between the zeros a_0^2 and a_1^2 in the same form as in Eq. (48).

First let us assume that Δ_D is negative, that is that r^2 defined in Eq. (29) is smaller than 1. Then the difference $a_1^2 - b_1^2$ that has to be positive is, by Eqs. (49c, d),

$$a_1^2 - b_1^2 = \frac{D_0 (N_1 - D_1)^2 + T_0 T_1}{(-T_1) (N_1 - D_1)}. \quad (50a)$$

The denominator in Eq. (50a) is positive since $-T_1$ and $N_1 - D_1$ are both positive in categories (a1) and (a3). The numerator of $(a_1^2 - b_1^2)/D_0$ is

$$(N_1 - D_1)^2 - (N_1 - D_1) T_0 D_1 / D_0 + T_0^2 / D_0. \quad (50b)$$

The quadratic polynomial in $N_1 - D_1$, presented in (50b), has the discriminant

$$\Delta = T_0^2 D_1^2 / D_0^2 - 4 T_0^2 / D_0 = \frac{T_0^2}{D_0^2} \Delta_D. \quad (50c)$$

Thus, if Δ_D is negative, Δ is also negative and the difference $a_1^2 - b_1^2$ is positive.

Secondly let us assume that Δ_D is positive, but Eq. (48d) does not hold; the zero b_1^2 in Eq. (48) is not located between the poles a_0^2 and a_1^2 . This is only possible when

$$b_1^2 < a_1^2 \text{ and } b_1^2 < a_0^2, \quad (50d)$$

since $D_1 = a_0^2 + a_1^2$ and $D_1 - b_1^2$ has to be positive. But then the pole b_1^2 in the function of type P_3 is right of the zero a_1^2 in this function and our assumption that the second component is of the type P_3 is true also in the second event.

There is another fact that we want to point out: Suppose the decomposition Eq. (44a) with $F_{ak}(s)$ of the type Q_4^{-1} is possible. Then all Eqs. (48a, ..., d) hold. By Eqs. (48a, b, c) we obtain

$$D_0 = a_0^2 a_1^2, \quad (51a)$$

$$D_1 = a_0^2 + a_1^2, \quad (51b)$$

$$T_0 = (N_1 - D_1) b_1^2. \quad (51c)$$

By Eq. (25)

$$T_1 = (N_1 - D_1) b_1^2 - (a_0^2 + a_1^2), \text{ and is negative by (48d).}$$

By Eq. (43c)

$$T_1' = (N_1 - D_1)^2 - T_1, \text{ and is positive when } T_1 \text{ is negative.}$$

Hence T_1' and $(-T_1)$ are positive.

Note that

$$N_0 D_1 - N_1 D_0 = T_0 D_1 - (N_1 - D_1) D_0 = (N_1 - D_1) (b_1^2 D_1 - D_0). \quad (51d)$$

In Eq. (51d) the difference $N_1 - D_1$ is positive. The second factor in this equation by Eqs. (51a, b)

$$b_1^2 D_1 - D_0 = a_1^2 (b_1^2 - a_0^2) + a_0^2 b_1^2$$

can only be positive when condition (48d) holds. Hence according to Figure 4a the function $F(s)$ must be of category (a1) with T_a negative. According to Table 3a the inverse function $F'(s) = 1/F(s)$ can be of category (b1) or (b4). Note that for this function

$$N_0' D_1' - N_1' D_0' = - (N_0 D_1 - N_1 D_0).$$

It follows by Figure 4b that only when $F'(s)$ is of category (b1), $N_0' D_1' - N_1' D_0'$ is exclusively negative, and $N_0 D_1 - N_1 D_0$ is exclusively positive. Therefore we can state:

If the function $F(s)$ is of category (a1) and the inverse function $F'(s) = 1/F(s)$ is of category (b1), then the functional component $F_{ak}(s)$ in the decomposition Eq. (44a) is of the type Q_4^{-1} . If $F(s)$ is of category (a1) and $F'(s)$ of category (b4), or if $F(s)$ is of category (a3) and $F'(s)$ necessarily is of category (b4), then the functional component $F_{ak}(s)$ in Eq. (44a) is of the type $1/(Q_1 + P_3)$.

4.2.2 THE FUNCTIONAL COMPONENT $F_{bk}(s)$ IN A DECOMPOSITION OF A FUNCTION $F(s)$ OF CATEGORIES (b1) OR (b3)

By Eq. (44b) we obtain

$$F_{bk}(s) = \frac{-T_0}{D_0} s \frac{s + (N_0 D_1 - N_1 D_0)/T_0}{s^2 + D_1 s + D_0}. \quad (52)$$

First we recognize that $F_{bk}(s)$ is no longer biquadratic. Let us find out whether it is pr. Since $F(s)$ is of the main-category (b), the test value T_0 is negative. Thus the factor in front of the fraction on the right side of Eq. (52) is positive. Consulting Figure 4b we find that in categories (b1) and (b3) the difference $N_0 D_1 - N_1 D_0$ is negative. Thus, with T_0 negative, the coefficient in the numerator of the fraction is positive. The function $F_{bk}(s)$ satisfies the postulation that all of its coefficients and the factor $(-T_0)/D_0$ are positive. We compute

$$\operatorname{Re} F_{bk}(j\omega) = \frac{\Omega}{D_0} \cdot \frac{T_a - \Omega T_0}{\Omega^2 + \Omega(D_1^2 - 2D_0) + D_0^2}. \quad (53)$$

We know that the denominator in Eq. (53) is positive for all positive Ω . According to Table 1 the test value T_a is positive in the considered categories (b1) and (b3). Hence, with the negative T_0 the numerator in Eq. (53) is also positive for all positive Ω , and consequently $F_{bk}(s)$ is pr.

According to Table 2 the square r^2 can be greater than 1 in both categories (b1) and (b3). Thus, if $r^2 > 1$, the denominator of $F_{bk}(s)$ can be factorized as in Eq. (48) and with

$$a_0^2 = \sqrt{D_0} (r - \sqrt{r^2 - 1}), \quad (54a)$$

$$a_1^2 = \sqrt{D_0} (r + \sqrt{r^2 - 1}), \quad (54b)$$

$$b_1^2 = (N_0 D_1 - N_1 D_0)/T_0, \quad (54c)$$

$$F_{bk}(s) = \frac{-T_0}{D_0} s \frac{s + b_1^2}{(s + a_0^2)(s + a_1^2)}. \quad (54)$$

Note that a_0^2 and a_1^2 are the same in Eqs. (48a, b) and (54a, b); but b_1^2 in Eq. (54c) is different from b_1^2 in Eq. (48c).

It is possible, but not necessary that

$$a_0^2 < b_1^2 < a_1^2. \quad (54d)$$

Then, with all Eqs. (54a, ..., d) true, the function in Eq. (54) is of the type

$$P_4^{-1} = \frac{s(s + b_1^2)}{k(s + a_0^2)(s + a_1^2)} \quad (\text{see Table A-3 in the appendix}),$$

for which

$$k = D_0/(-T_0). \quad (54e)$$

If $r^2 > 1$, but Eq. (54d) does not hold, or if $r^2 < 1$, then let us, for merely formal reasons, name the former b_1^2 now a_1^2 , thus defining

$$a_1^2 = (N_0 D_1 - N_1 D_0)/T_0, \quad (55a)$$

and let us propose the decomposition

$$\begin{aligned} \frac{1}{F_{bk}(s)} &= \frac{D_0(D_0 + sD_1 + s^2)}{s(-T_0)(a_1^2 + s)} \\ &= \frac{D_0}{s(-T_0)} \left[\frac{D_0}{a_1^2} + s \frac{s + b_1^2}{s + a_1^2} \right] \end{aligned} \quad (55)$$

where

$$b_1^2 = D_1 - D_0/a_1^2. \quad (55b)$$

Then

$$\frac{1}{F_{bk}(s)} = \frac{1}{k_1 s} + \frac{s + b_1^2}{k_2(s + a_1^2)} \quad (56)$$

where

$$k_1 = -(N_0 D_1 - N_1 D_0)/D_0^2, \quad (55c)$$

and

$$k_2 = (-T_0)/D_0. \quad (55d)$$

Note that b_1^2 in Eq. (54c) equals a_1^2 in Eq. (55a), but b_1^2 in Eq. (55b) is of course different from b_1^2 in Eq. (54c).

The first component in Eq. (56) is a function of the type

$$Q_1^{-1} = 1/k_1 s \quad (\text{see Table A-2 in the appendix}).$$

The second component in Eq. (56) is supposed to be a function of the type

$$P_3^{-1} = (s + b_1^2)/k_2 (s + a_1^2) \quad (\text{see Table A-3 in the appendix}).$$

We have to prove that the second functional component in Eq. (56) is a function of the type P_3^{-1} where $a_1^2 > b_1^2$ or $a_1^2 - b_1^2$ is positive. For this purpose let us compute

$$a_1^2 - b_1^2 = (N_0 D_1 - N_1 D_0)/T_0 - D_1 + D_0 T_0/(N_0 D_1 - N_1 D_0),$$

hence

$$T_0 (N_0 D_1 - N_1 D_0) (a_1^2 - b_1^2) = (N_0 D_1 - N_1 D_0)^2 - (N_0 D_1 - N_1 D_0) D_1 T_0 + D_0 T_0^2.$$

The left side of this equation is positive when $(a_1^2 - b_1^2)$ is positive, since in categories (b1) and (b3) the test value T_0 and the difference $N_0 D_1 - N_1 D_0$ are both negative. Thus we have to prove that the right side of this equation is positive.

The right side is a quadratic polynomial in $N_0 D_1 - N_1 D_0$. Its discriminant is

$$\Delta = D_1 T_0^2 - 4 D_0 T_0^2 = T_0^2 (D_1 - 4 D_0) = T_0^2 \Delta_D,$$

with Δ_D defined in Eq. (47). Thus Δ has the same polarity as Δ_D . If $r^2 < 1$ then Δ and Δ_D are negative and then the right side is certainly positive. This is true in particular when $F(s)$ is of category (b3) where $r^2 < 0.5$.

Now assume that $r^2 > 1$ and that then in Eq. (54) the zero b_1^2 is not between the poles a_0^2 and a_1^2 ; but remember that for Eqs. (55), (55a, ..., d), and (56) we redefined b_1^2 as a_1^2 . With a_0^2 , a_1^2 , and b_1^2 defined in Eqs. (54a, b, c), $T_a = T_0 (D_0 - b_1^2 D_1)$ and has to be positive with negative T_0 in categories (b1) and (b3). Thus

$$D_1 b_1^2 - D_0 = (a_0^2 + a_1^2) b_1^2 - a_0^2 a_1^2$$

has to be positive. This is ensured when b_1^2 is not between a_0^2 and a_1^2 if $b_1^2 > a_1^2$. Thus, when the poles and zeros in Eq. (54) do not alternate, the zero

b_1^2 has to be located left to the pole a_1^2 and then our assumption that the second component in Eq. (56) is of the type P_3^{-1} is correct.

Similarly, as in the discussion of the function $F_{ak}(s)$ we can also find that a function $F_{bk}(s)$ of the type P_4^{-1} can only be obtained when $F(s)$ is of category (b1) and $F'(s) = 1/F(s)$ is of category (a1). Otherwise $1/F_{bk}(s)$ is of the type presented in Eq. (56). Thus we state:

If the function $F(s)$ is of category (a1) and the inverse function $F'(s) = 1/F(s)$ is of category (a1), then the functional component $F_{bk}(s)$ in the decomposition Eq. (44b) is of the type P_4^{-1} . If $F(s)$ is of category (b1) and $F'(s)$ of category (a4), or if $F(s)$ is of category (b3) and $F'(s)$ necessarily is of category (a4), then the functional component $F_{bk}(s)$ in Eq. (44b) is of the type $1/(Q_1^{-1} + P_3^{-1})$.

4.2.3 THE FUNCTIONAL COMPONENT $(1-N_m)\bar{F}(s)$ IN A DECOMPOSITION OF A FUNCTION $F(s)$ OF CATEGORIES (a2), (a4), (b2), OR (b4)

By Eq. (44c) we obtain

$$\bar{F}(s) = (F(s) - N_m)/(1 - N_m) = \frac{s^2 + \bar{N}_1 s + \bar{N}_0}{s^2 + \bar{D}_1 s + \bar{D}_0}, \quad (57)$$

where since $F(s)$ is a singular function

$$\bar{N}_1 \bar{D}_1 = (\sqrt{\bar{N}_0} - \sqrt{\bar{D}_0})^2. \quad (57a)$$

The constant $N_m < 1$ is obtained by Eq. (28) where depending on the category of $F(s)$ the frequency Ω_m is obtained by Eq. (26) for categories (a2) and (b2) and by Eq. (27) for categories (a4) and (b4). In Eq. (57)

$$\bar{D}_0 = D_0, \quad (57b)$$

$$\bar{D}_1 = D_1, \quad (57c)$$

$$\bar{N}_0 = (N_0 - N_m D_0)/(1 - N_m), \quad (57d)$$

$$\bar{N}_1 = (N_1 - N_m D_1)/(1 - N_m). \quad (57e)$$

It is necessary to compute N_m with high accuracy; if the accuracy is not sufficient, a considerable error in Eq. (57a) will occur even though the computation does not involve any other error. As the tool for the computations we used a FRIDEN desk calculator Model SRQ 10. In dealing with biquadratic functions it is sufficient to get the results to 6-figure accuracy to the right of the decimal point. Equation (57a) has been more than sufficiently satisfied by increasing the accuracy to 9 figures to

the right of the decimal point for the computation of N_m and \bar{N}_0, \bar{N}_1 . If the accuracy is not sufficient, then N_m can be corrected in the following way:

According to Eq. (57a) the constant N_m must satisfy the quadratic equation

$$A_2 N_m^2 + A_1 N_m + A_0 = 0, \quad (58)$$

in which

$$A_2 = D_1^2 (D_1^2 - 4D_0), \quad (58a)$$

$$A_1 = -2 (D_1^2 - 2D_0) (N_1 D_1 - N_0 - D_0) - 2D_0 (N_0 + D_0), \quad (58b)$$

$$A_0 = (N_1 D_1 - N_0 - D_0)^2 - 4N_0 D_0. \quad (58c)$$

An approximation of N_m is known by the insufficiency. This approximation can be improved by instructions given in (Haase, 1963b).

There is an alternative which at first glance may seem to be very trivial: we can also write Eq. (44c) as

$$F(s) = N_m + \frac{1}{(1 - N_m)' \bar{F}'(s)}, \quad (59)$$

when we define

$$(1 - N_m)' = 1 / (1 - N_m), \quad (59a)$$

and

$$\bar{F}'(s) = 1 / \bar{F}(s). \quad (59b)$$

Then by Eq. (59)

$$\bar{F}'(s) = \frac{1 - N_m}{(F(s) - N_m)} = \frac{s^2 + \bar{N}_1' s + \bar{N}_0'}{s^2 + \bar{D}_1' s + \bar{D}_0'}, \quad (60)$$

where also $\bar{F}'(s)$ is a singular function so that

$$\bar{N}_1' \bar{D}_1' = (\sqrt{\bar{N}_0'} - \sqrt{\bar{D}_0'})^2. \quad (60a)$$

The coefficients in Eq. (60) are

$$\bar{D}'_0 = (N_0 - N_m D_0) / (1 - N_m), \quad (60b)$$

$$\bar{D}'_1 = (N_1 - N_m D_1) / (1 - N_m), \quad (60c)$$

$$\bar{N}'_0 = D_0, \quad (60d)$$

$$\bar{N}'_1 = D_1. \quad (60e)$$

The difference between the decompositions Eq. (44c) and Eq. (59) is mentioned in the next section when we interpret the immittances. By Eq. (44c) for instance

$$\text{Impedance } F(s) = \text{Impedance } N_m + \text{Impedance } (1 - N_m) \bar{F}(s),$$

and by Eq. (59)

$$\text{Impedance } F(s) = \text{Impedance } N_m + \frac{1}{\text{Admittance } (1 - N_m) \bar{F}'(s)}.$$

Similarly by Eq. (44c)

$$\text{Admittance } F(s) = \text{Admittance } N_m + \text{Admittance } (1 - N_m) \bar{F}(s),$$

and by Eq. (59)

$$\text{Admittance } F(s) = \text{Admittance } N_m + \frac{1}{\text{Impedance } (1 - N_m) \bar{F}'(s)}.$$

We have shown in Tables 3a, b that a non-singular function that is of the category (a4) or (b4) allows inverse functions $\bar{F}'(s)$ that can be of any of the four sub-categories in the other main-category. This does not hold for singular functions $\bar{F}(s)$ and $\bar{F}'(s)$. These functions can only be either of sub-category (2) or (4). Thus for singular functions Tables 4a, b are true:

Table 4a. Categories of $\bar{F}'(s)$ for Certain Categories of $\bar{F}(s)$

If $\bar{F}(s)$ is of category	then $\bar{F}'(s) = 1/\bar{F}(s)$ is of either one of the categories
(a2)	(b2) or (b4)
(a4)	(b2) or (b4)

Table 4b. Categories of $\bar{F}(s)$ for Certain Categories of $\bar{F}'(s)$

If $\bar{F}(s)$ is of category	then $\bar{F}'(s) = 1/\bar{F}(s)$ is of either one of the categories
(b2)	(a2) or (a4)
(b4)	(a2) or (a4)

It also follows from Tables 4a, b that a singular function $\bar{F}(s)$ cannot be of an odd sub-category (1) or (3).

When the function $F(s)$ is of an even (odd) sub-category and the function $F'(s)$ is of an odd (even) sub-category, the decomposition of $F(s)$ is clearly distinct from that of $F'(s)$. The one has the functional component $F_{ak}(s)$ or $F_{bk}(s)$, the other the component $(1-N_m)\bar{F}(s)$. But when $F(s)$ and $F'(s)$ are both of even sub-categories, then both functions have the functional component $(1-N_m)\bar{F}(s)$ in their decomposition. In order to avoid too many notations, we will agree that in this event we decompose $F(s)$ as it is, and by the decomposition we will obtain one set of N_m , $(1-N_m)$, $(1-N_m)'$, $\bar{F}(s)$, and $\bar{F}'(s)$. Then we decompose $F(s)$ with D 's interchanged with N 's and get a second set of decomposition constants and functionals.

Consider now the singular function $\bar{F}(s)$ in the notation of Eq. (57). Such a function can be written in the following form:

$$\bar{F}(s) = \frac{vx(n-1)^2\phi(s)\Phi(s) + vzn^2\phi(s)\zeta(s) + xzn\Phi(s)\zeta(s)}{v\phi(s) + xn\Phi(s) + zn\zeta(s)}. \quad (61)$$

In Eq. (61) v and n are real constants of the same sign polarity (when v is positive n is also positive, and when v is negative n is also negative); x and z are real and always positive constants. The notations $\phi(s)$, $\Phi(s)$, and $\zeta(s)$ are normalized positive real-frequency functions such that one of them is s , another

one $1/s$, and the third one 1 . There are six possible variations for the distribution of these particular frequency functions over $\phi(s)$, $\Phi(s)$, and $\zeta(s)$ as we show in Table 5:

Table 5. The Six Possible Distributions of s , $1/s$, and 1 over $\phi(s)$, $\Phi(s)$, and $\zeta(s)$

Function Notation	Selections		
\bar{F}_a	$\phi(s) = s,$	$\Phi(s) = 1/s,$	$\zeta(s) = 1,$
\bar{F}_{ax}		$\Phi(s) = 1,$	$\zeta(s) = 1/s,$
\bar{F}_b	$\phi(s) = 1/s,$	$\Phi(s) = s,$	$\zeta(s) = 1,$
\bar{F}_{bx}		$\Phi(s) = 1,$	$\zeta(s) = s,$
\bar{F}_c	$\phi(s) = 1,$	$\Phi(s) = s,$	$\zeta(s) = 1/s,$
\bar{F}_{cx}		$\Phi(s) = 1/s,$	$\zeta(s) = s.$

The functions \bar{F}_a and \bar{F}_{ax} have the selection $\phi(s) = s$ in common; the selections of the functions $1/s$ and 1 are interchanged. The situations are similar in the function pairs \bar{F}_b , \bar{F}_{bx} and \bar{F}_c , \bar{F}_{cx} . We want that

$$\bar{F}_a = \bar{F}_{ax} = \bar{F}_b = \bar{F}_{bx} = \bar{F}_c = \bar{F}_{cx} = \bar{F}(s). \quad (62)$$

Substituting the selections according to Table 5 we obtain:

$$\bar{F}_a = zn^2 \frac{s^2 + sx(n-1)^2/zn^2 + x/vn}{s^2 + szn/v + xn/v}, \quad (63)$$

where

$$zn^2 = 1. \quad (63a)$$

$$\bar{F}_{ax} = x(n-1)^2 \frac{s^2 + szn^2/x(n-1)^2 + zn/v(n-1)^2}{s^2 + sxn/v + zn/v}, \quad (64)$$

where

$$x(n-1)^2 = 1. \quad (64a)$$

$$\bar{F}_b = z \frac{s^2 + sv(n-1)^2/zn + vn/x}{s^2 + sz/x + v/xn}, \quad (65)$$

where

$$z = 1. \quad (65a)$$

$$\bar{F}_{bx} = x \frac{s^2 + svn/x + v(n-1)^2/zn}{s^2 + sx/z + v/zn} \quad (66)$$

where

$$x = 1. \quad (66a)$$

$$\bar{F}_c = \frac{v(n-1)^2}{n} \cdot \frac{s^2 + szn/v(n-1)^2 + zn^2/x(n-1)^2}{s^2 + sv/xn + z/x}, \quad (67)$$

where

$$\frac{v(n-1)^2}{n} = 1. \quad (67a)$$

$$\bar{F}_{cx} = vn \frac{s^2 + sx/vn + x(n-1)^2/zn^2}{s^2 + sv/zn + x/z}, \quad (68)$$

where

$$vn = 1. \quad (68a)$$

Equations (63a, ..., 68a) follow from the fact that $\bar{F}(s) = 1$ for $s \rightarrow \infty$.

Since all the function variations are the same according to Eq. (62) we can make comparisons between the coefficients N_0, D_0, N_1, D_1 and the respective expressions in Eqs. (63, ..., 68) and we thus obtain the results in Table 6.

Table 6. The Constants n, u, v, w, x, z as Functions of the Coefficients D_0, D_1, N_0, N_1 for \bar{F}_a, b, c and $\bar{F}_{ax, bx, cx}$

	F_a	F_{ax}	F_y	F_{bx}	F_c	F_{cx}
n	$\sqrt{\frac{D_0}{N_0}}$	$\pm \sqrt{\frac{N_1 D_1}{N_0}}$	$\sqrt{\frac{N_0}{D_0}}$	$\pm \sqrt{\frac{N_1 D_1}{D_0}}$	$\pm \sqrt{\frac{N_0}{N_1 D_1}}$	$\pm \sqrt{\frac{D_0}{N_1 D_1}}$
u	$\pm \sqrt{\frac{N_1}{D_1 D_0}}$	\rightarrow	$\pm \sqrt{\frac{N_1 N_0}{D_1}}$	\rightarrow	$\sqrt{\frac{N_0}{D_0}}$	\rightarrow
v	$\frac{1}{D_1} \sqrt{\frac{N_0}{D_0}}$	\rightarrow	$\frac{1}{D_1} \sqrt{\frac{N_0 D_0}{D_1}}$	\rightarrow	$\pm \frac{1}{D_0} \sqrt{\frac{N_1 N_0 D_1}{D_0}}$	\rightarrow
w	$\pm \frac{1}{D_0} \sqrt{\frac{N_1 N_0}{D_1}}$	\rightarrow	$\pm \sqrt{\frac{N_1 D_0}{D_1}}$	\rightarrow	$\pm \sqrt{\frac{N_1 D_1}{D_0}}$	\rightarrow
x	N_0/D_1	\rightarrow	$1/D_1$	\rightarrow	N_1/D_0	\rightarrow
z	N_0/D_0	\rightarrow	1	\rightarrow	N_1	\rightarrow
Upper Lower	(a) of $F(s)$, no sign means + signs for main category (b)		The formulas also hold for N'_0, N'_1, D'_0, D'_1 , and $F'(s)$.			

Table 6 also contains the constants

$$u = v(n - 1) \quad (69)$$

and

$$w = v\left(\frac{1}{n} - 1\right). \quad (70)$$

These constants will be of interest later. But note that

$$1/u + 1/v + 1/w = 0, \quad (71)$$

which makes it necessary that one of the u, v, w be negative, whereas the other two are positive in the sign polarity. Thus, when v and n are negative, then u and w must both be positive. But when v and n are positive, then either u or w must be negative. The sign polarities depend only on the main-category of $\bar{F}(s)$ which is the same as the main-category of $F(s)$.

Table 6 also shows that although when we want to obtain the constants n, u, v, w, x, z of all six variations, only half of them have to be computed. The constant u is the same for \bar{F}_a and \bar{F}_{ax} , for \bar{F}_b and \bar{F}_{bx} , and for \bar{F}_c and \bar{F}_{cx} . The constants v and w interchange between these pairs, also the constants x and z . As is indicated by the corners of the triangles \triangle and ∇ the respective constants n are vice versa and inverse. But note also that

$$n \text{ in } \bar{F}_a = 1/u \text{ in } \bar{F}_c, \quad (72a)$$

$$n \text{ in } \bar{F}_b = u \text{ in } \bar{F}_c, \quad (72b)$$

$$n \text{ in } \bar{F}_c = -u/w \text{ in } \bar{F}_a. \quad (72c)$$

Also:

$$n \text{ in } \bar{F}_{ax} = -w/u \text{ in } \bar{F}_c, \quad (73a)$$

$$n \text{ in } \bar{F}_{bx} = w \text{ in } \bar{F}_c, \quad (73b)$$

$$n \text{ in } \bar{F}_{cx} = 1/w \text{ in } \bar{F}_c. \quad (73c)$$

Thus when the constants u and w in \bar{F}_c are known, the constant n can be easily computed for all variations of $\bar{F}(s)$.

Table 6 also holds, of course, for the inverse function $\bar{F}'(s)$ when the coefficients $\bar{D}_0, \bar{D}_1, \bar{N}_0, \bar{N}_1$ are replaced by the coefficients $\bar{D}'_0, \bar{D}'_1, \bar{N}'_0, \bar{N}'_1$ as defined in Eqs. (60b, ..., e).

For convenience we give in Table 7 references to the respective equations to be used when decomposing the function $F(s)$.

Table 7. Decomposition Table

$F(s)$ of Category (a1) or (a3) Decomposition $F(s) = 1 + F_{ak}(s)$	
$F(s)$ of (a1)	$F_{ak}(s)$ of type Q_4^{-1}
$F'(s)$ of (b1)	see Eqs. (48) and (48a, ..., e)
$F(s)$ of (a1) or (a3)	$F_{ak}(s)$ of type $1/(Q_1 + P_1)$
$F'(s)$ of (b4) or (b4)	see Eqs. (49) and (49a, ..., d)
$F(s)$ of Category (b1) or (b3) Decomposition $F(s) = N_0/D_0 + F_{bk}(s)$	
$F(s)$ of (a1)	$F_{bk}(s)$ of type P_4^{-1}
$F'(s)$ of (a1)	see Eqs. (54) and (54a, ..., e)
$F(s)$ of (b1) or (b3)	$F_{bk}(s)$ of type $1/(Q_1^{-1} + P_3^{-1})$
$F'(s)$ of (a4) or (a4)	see Eqs. (55) and (55a, ..., d)
$F(s)$ of Category (a2), (b2), (a4), (b4) Decomposition $F(s) = N_m + (1-N_m)\bar{F}(s) = N_m + (1-N_m)/\bar{F}'(s)$	
$F(s)$ and $\bar{F}(s)$ of (a2) or (b2)	Get N_m by Eqs. (26) and (26a)
$F(s)$ and $\bar{F}(s)$ of (a4) or (b4)	Get N_m by Eqs. (27) and (27a)
$F(s)$ and $\bar{F}(s)$ of (a2), (b2), (a4), (b4)	Get coefficients $\bar{D}_0, \bar{D}_1, \bar{N}_0, \bar{N}_1$ of $\bar{F}(s)$ by Eqs. (57b, ..., e)
	Get variations $\bar{F}_a, \dots, \bar{F}_{cx}$ directly by Table 6
$\bar{F}'(s)$ of (a2), (b2), (a4), (b4)	Get coefficients $\bar{D}'_0, \bar{D}'_1, \bar{N}'_0, \bar{N}'_1$ of $\bar{F}'(s)$ by Eqs. (60b, ..., e)
	Get variations $\bar{F}'_a, \dots, \bar{F}'_{cx}$ by Table 6 with all capitals primed.

5. REALIZATIONS OF DRIVING-POINT IMMITTANCE FUNCTIONS $F(s)$

By "realizing" an immittance function we understand that we are looking for a network structure such that the driving-point immittance measured at its terminals (see Figure 1) is $F(s)$. If we propose $F(s)$, we will always obtain two classes of networks: one class has the driving-point impedance identical with $F(s)$, the other has the driving-point admittance identical with $F(s)$. Each class usually has several equivalent networks. But each network in the one class has a dual network in the other. When the networks are of the ladder type, then the dual network is found simply by drawing the graphic representation of one network; then by drawing the dual graph representation (each series branch becomes a shunt branch and vice versa). Both graphs are topologically related. Then the element combinations in the first graph have to be replaced by the dual element combinations for each topologically related branch; this means that if each branch implies only one element, for instance, a resistance, an inductance, a capacitance in the first graph become a conductance, a capacitance, an inductance in the second graph.

Practically, however, the situation is usually such that we are confronted with the problem of realizing a network that has a prescribed impedance function $Z(s)$ or a network that has a prescribed admittance function $\Gamma(s)$, where $Z(s)$ and respectively $\Gamma(s)$ are pr biquadratic functions. Thus $Z(s)$ or $\Gamma(s)$ is proposed and not $F(s)$. Hence we can adjust $F(s)$ to the proposed function. If $Z(s)$ is proposed we make $Z(s) \equiv F(s)$; but then necessarily $\Gamma(s) \equiv F'(s)$ and these two identities yield two classes of networks which of course are not dual; but each network in the one class may have some structural similarities with a network in the other class. Similarly, when $\Gamma(s)$ is proposed, we make $\Gamma(s) \equiv F(s)$ and then $Z(s) \equiv F'(s)$.

5.1 The Impedance and the Admittance Interpretation of $F(s)$

As a neutral concept the immittance function $F(s)$ can be deliberately interpreted as an impedance $Z(s)$ or as an admittance $\Gamma(s)$, or the immittance function $F(s)$ can be deliberately adjusted to an impedance $Z(s)$ or an admittance $\Gamma(s)$. It makes no difference whether we stick to the first or to the second formulation. In the third section we have decomposed $F(s)$ in two or more components and we determined that these components are pr and of lower rank ($F_{ak}(s)$ and $F_{bk}(s)$ and the constants 1 and N_0/D_0 respectively), or we have determined that the functional component implied a singular function whereas the non-singular component was non singular. Of course, when $F(s)$ is interpreted in one way or the other, the interpretation also holds for the components. Say, we interpret $F(s) \equiv Z(s)$ as an impedance, and assume that we decomposed

$$F(s) = F_1(s) + F_2(s), \quad (74)$$

where $F_1(s)$ and $F_2(s)$ are pr. Then

$$\text{impedance } F(s) = \text{impedance } F_1(s) + \text{impedance } F_2(s). \quad (74a)$$

But with

$$1/F_1(s) = F'_1(s) \text{ and } 1/F_2(s) = F'_2(s)$$

we can write Eq. (74a) also in the forms

$$\text{impedance } F(s) = \frac{1}{\text{admittance } F'_1(s)} + \frac{1}{\text{admittance } F'_2(s)}, \quad (74b)$$

$$= \text{impedance } F_1(s) + \frac{1}{\text{admittance } F'_2(s)}, \quad (74c)$$

$$= \frac{1}{\text{admittance } F'_1(s)} + \text{impedance } F_2(s). \quad (74d)$$

When a component $F_j(s)$ is of the type P_i , Q_i , P_i^{-1} , or Q_i^{-1} as listed in the appendix, then impedance $F_j(s)$ equals exactly $1/\text{admittance } F'_j(s)$ since $P_i = 1/P_i^{-1}$ and $Q_i = 1/Q_i^{-1}$. (Note that a constant can be considered as a function of the type P_1 , Table A-2 or as a function of the type P_1^{-1} , Table A-3.) Hence there is also no difference in the circuits realizing impedance $F_j(s)$ or $1/\text{admittance } F'_j(s)$. But if $F_j(s) = (1-N_m)\bar{F}(s)$, then a circuit realizing the impedance $F_j(s)$ is different from a circuit realizing $1/\text{admittance } F'_j(s)$ as we will see in Section 5.2.3.

5.2 RLC Circuits Realizing $F(s)$ in Both Interpretations

Since we know the decomposition of $F(s)$ as compiled in Table 7, we know also its realization as soon as the realizations of the decomposition components are known. The decompositions are sums. In the impedance interpretation the component circuits have to be linked in series, in the admittance interpretation they have to be linked in parallel. As far as the components are of the types P_i , Q_i or their reciprocals, their realizations are communicated in Haase (1963a) with the respective tables reprinted in the appendix. Realizations of $\bar{F}(s)$ will be given in Section 5.2.3.

5.2.1 REALIZATIONS OF IMMITTANCE FUNCTIONS $F(s)$ OF THE CATEGORIES (a1) AND (a3)

An immittance function $F(s)$ that is of categories (a1) or (a3) has the decomposition of Eq. (44a). The first component 1 in the decomposition is a resistance $R_0 = 1$ in both interpretations of $F(s)$. Block realizations for the impedance interpretation of $F(s)$ are shown in Table 8a; for the admittance interpretation they are shown in Table 8b. If $F_{ak}(s)$ is of the type Q_4^{-1} , then an impedance Q_4^{-1} is in series with $R_0 = 1$ in the impedance interpretation (see upper part of Table 8a) and an admittance Q_4^{-1} is in parallel with $R_0 = 1$ in the admittance interpretation (see upper part of Table 8b). As it is shown in the lower part of Tables 8a, b, if $F_{ak}(s)$ is of the type $1/(Q_1 + P_3)$, then an admittance $(Q_1 + P_3)$ is in series with R_0 in the impedance interpretation, and an impedance $(Q_1 + P_3)$ is in parallel with R_0 in the admittance interpretation. The tables show only part of the realization possibilities, since we are also able to realize $1/F(s)$. But for these realizations we do not have as yet sufficient information since $1/F(s)$ is of category (b1) or (b4). We also want to postpone going into the detail of the box structure.

5.2.2 REALIZATIONS OF IMMITTANCE FUNCTIONS $F(s)$ OF THE CATEGORIES (b1) AND (b3)

An immittance function $F(s)$ that is of categories (b1) or (b3) has the decomposition Eq. (44b). The first component N_0/D_0 in the decomposition is a resistance $R_0 = N_0/D_0$ in the impedance interpretation and a resistance $R_0 = D_0/N_0$ in the admittance interpretation. In the latter interpretation we could also call it a conductance $G_0 = N_0/D_0$; this would be more systematical, but we prefer to use only the term "resistance." Block realizations for the impedance interpretation of $F(s)$ are shown in Table 9a, for the admittance interpretation in Table 9b. If $F_{bk}(s)$ is of the type P_4^{-1} , then an impedance P_4^{-1} is in series with $R_0 = N_0/D_0$ in the impedance interpretation of $F(s)$ (see upper part of Table 9a), and an admittance P_4^{-1} is in parallel with $R_0 = D_0/N_0$ in the admittance interpretation of $F(s)$ (see upper part of Table 9b). As shown in the lower parts of Tables 9a, b, if $F_{bk}(s)$ is of the type $1/(Q_1^{-1} + P_3^{-1})$, in the impedance interpretation an admittance $1/(Q_1^{-1} + P_3^{-1})$ is in series with R_0 , and in the admittance interpretation an impedance $1/(Q_1^{-1} + P_3^{-1})$ is in parallel with R_0 in the admittance interpretation. Also these tables do not exhaust all of the possibilities of realizing $F(s)$, since we can also realize $1/F(s)$. This, however, together with the information about the circuit structures in the boxes, we will postpone.

Table 8a. Block Realizations of $Z(s) \equiv F(s)$ of Categories (a1) and/or (a3)

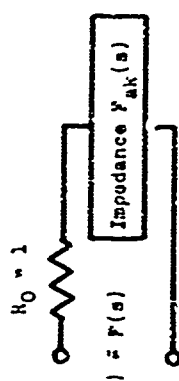
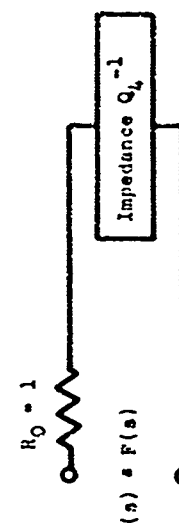

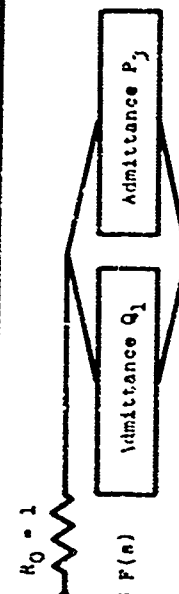
Driving - Point Impedance $Z(s) \equiv F(s)$ of Categories (a1) and (a3)	
<p> $R_0 = 1$ $Z(s) \equiv F(s)$  $Z(s) \equiv F(s)$ Series Combination of Resistance and Impedance </p>	<p> $R_0 = 1$ $Z(s) \equiv F(s)$  $Z(s) \equiv F(s)$ F(s) of Category (a1) $1/F(s)$ of Category (b1) </p>
<p> $R_0 = 1$ $Z(s) \equiv F(s)$  $Z(s) \equiv F(s)$ Series Combination of Resistance and Inverse Admittance </p>	<p> $R_0 = 1$ $Z(s) \equiv F(s)$  $Z(s) \equiv F(s)$ F(s) of Category (a1) or (a3) $1/F(s)$ of Category (b4) </p>

Table 8b. Block Realizations of $\Gamma(s) \equiv F(s)$ of Categories (a1) and/or (a3)

Driving - Point Admittance Function $\Gamma(s) \equiv F(s)$ of Categories (a1) and (a3)	
<p> $\Gamma(s) \equiv F(s)$ $R_0 = 1$ Admittance $F_{ak}(s)$ </p> <p>Parallel Combination of Resistance and Admittance</p>	<p> $\Gamma(s) \equiv F(s)$ $R_0 = 1$ Admittance ζ_k^{-1} </p> <p>$F(s)$ of Category (a1) $1/F(s)$ of Category (b1)</p>
<p> $\Gamma(s) \equiv F(s)$ $R_0 = 1$ Impedance $1/F_{ak}(s)$ </p> <p>Parallel Combination of Resistance and Inverse Impedance</p>	<p> $\Gamma(s) \equiv F(s)$ $R_0 = 1$ Impedance C_1 Impedance P_3 Impedance P_3 </p> <p>$F(s)$ of Category (a1) or (a3) $1/F(s)$ of Category (b4)</p>

Table 9a. Block Realizations of $Z(s) \equiv F(s)$ of Categories (b1) and/or (b3)

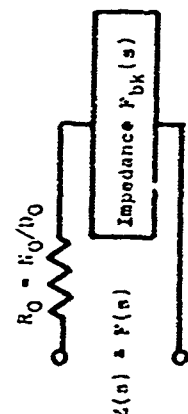
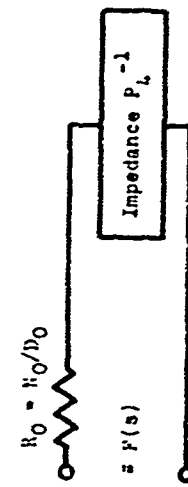
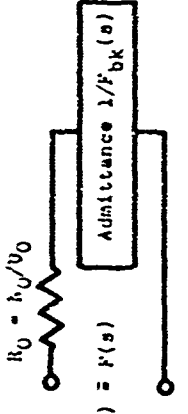
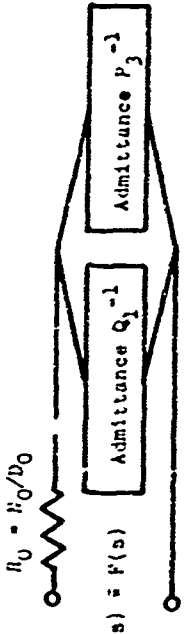
<p>Driving - Point Impedance $Z(s) \equiv F(s)$ of Categories (b1) and (b3)</p>  <p>$Z(s) \equiv F(s)$</p> <p>Series Combination of Resistance and Impedance</p>	 <p>$Z(s) \equiv F(s)$</p> <p>$F(s)$ of Category (b1) $1/F(s)$ of Category (a1)</p>
 <p>$Z(s) \equiv F(s)$</p> <p>Series Combination of Resistance and Inverse Admittance</p>	 <p>$Z(s) \equiv F(s)$</p> <p>$F(s)$ of Category (b1) or (b3) $1/F(s)$ of Category (a4)</p>

Table 9b. Block Realizations of $\Gamma(s) \equiv F(s)$ of Categories (b1) and/or (b3)

Driving-Point Admittance Function $\Pi(s) \equiv F(s)$ of Categories (b1) and (b3)	
<p> $\Pi(s) \equiv F(s)$ $R_0 = D_0/h_0$ Admittance $F_{bk}(s)$ </p> <p>Parallel Combination of Resistance and Admittance</p>	<p> $\Pi(s) \equiv F(s)$ $R_0 = D_0/h_0$ Admittance P_L^{-1} </p> <p>$F(s)$ of Category (b1) $1/F(s)$ of Category (a1)</p>
<p> $\Pi(s) \equiv F(s)$ $R_0 = D_0/h_0$ Impedance $1/F_{bk}(s)$ </p> <p>Parallel Combination of Resistance and Inverse Impedance</p>	<p> $\Pi(s) \equiv F(s)$ $R_0 = D_0/h_0$ Impedance ζ_1^{-1} Impedance P_3^{-1} </p> <p>$F(s)$ of Category (b1) or (b3) $1/F(s)$ of Category (a4)</p>

5.2.3 REALIZATIONS OF IMMITTANCE FUNCTIONS $F(s)$ OF THE CATEGORIES (a2), (b2), (a4), OR (b4)

An immittance function that is of any of the categories (a2), (b2), (a4), or (b4) has the decomposition Eq. (44c). The first component $N_m < 1$ in the decomposition is a resistance $R_0 = N_m$ in the impedance interpretation and a resistance $R_0 = 1/N_m$ in the admittance interpretation. Again we prefer to avoid the conductance term $G_0 = N_m$ in the latter interpretation. Block realizations for the impedance interpretation of $F(s)$ are shown in Table 10a and for the admittance interpretation of $F(s)$ in Table 10b.

For the impedance interpretation in Table 10a we find one structure where a box one-port with the driving-point impedance $(1-N_m) \bar{F}(s)$ is in series with the resistance $R_0 = N_m$ and another one where the box one-port has the driving-point admittance $(1-N_m) \bar{F}'(s)$. For the admittance interpretation in Table 10b we also find two arrangements: in one a box one-port that has the driving-point admittance $(1-N_m) \bar{F}(s)$ is in parallel with the resistance $R_0 = 1/N_m$, in the other a box one-port, with the driving-point impedance $(1-N_m) \bar{F}'(s)$, is in parallel with this resistance. Let us forget for a moment the numerical factors $(1-N_m)$ and $(1-N_m)'$ and let us ask: is there a circuit structure that has $\bar{Z}(s) \equiv \bar{F}(s)$ as driving-point impedance, or another structure that has $\bar{Y}(s) \equiv \bar{F}(s)$ as driving-point admittance? The function $\bar{F}(s)$ in these interpretations is a normalized and singular biquadratic function. We can restrict ourselves to the function $\bar{F}(s)$, since $\bar{F}'(s)$ is nothing more than a formality in the notation.

In fact, there is such a structure. In the impedance interpretation of $\bar{F}(s)$ it makes use of "a perfectly coupled and shunt-augmented T" that has been discussed in an earlier paper of the author (Haase, 1965). In the admittance interpretation of $\bar{F}(s)$ "a perfectly coupled and shunt-augmented Pi" has to be used instead, since the admittance interpretation is the dual of the impedance interpretation. At this point let us discuss the realizations of $\bar{F}(s)$ in a structure where we represent the branch immittances by boxes.

Consider the one-port shown in Figure 6a. It consists of a T that is terminated by the branch Z. The T has the series branches U and W and a shunt branch that is composed of the branch elements V and X. The branches U, V, W make a coupled unit, thus we may say that X is an augmentation in the shunt branch of the T. Let all capital notations in Figure 6a be impedances. Let further

$$1/U + 1/V + 1/W = 0. \quad (75)$$

Then the driving-point impedance of the terminated T in Figure 6a is

$$\bar{Z}(s) = \frac{X(U + W) + Z(U + V) + XZ}{(V + W) + (X + Z)}. \quad (76)$$

Table 10a. Block Realizations of $Z(s) = F(s)$ of Categories (a2), (a4), (b2) and/or (b4)

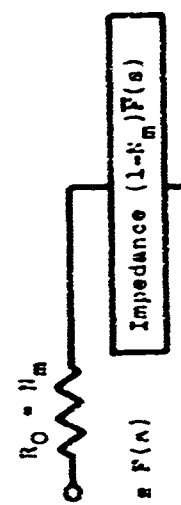
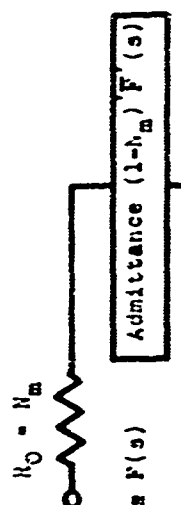
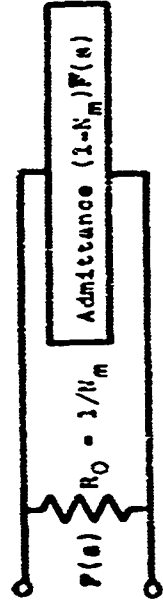
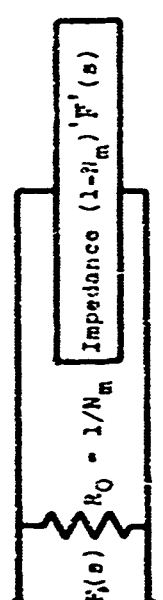
Driving - Point Impedance $Z(s) = F(s)$ of Categories (a2), (a4), (b2), (b4)	
$R_0 = 1/N_m$ $Z(s) = F(s)$  Series Combination of Resistance and Impedance	$R_0 = N_m$ $Z(s) = F(s)$  Series Combination of Resistance and Inverse Admittance

Table 10b. Block Realizations of $\Gamma(s) = F(s)$ of Categories (a2), (a4), (b2) and/or (b4)

Driving - Point Admittance $\Gamma(s) = F(s)$ of Categories (a2), (a4), (b2), (b4)	
$R_0 = 1/N_m$ $\Gamma(s) = F(s)$  Parallel Combination of Resistance and Admittance	$R_0 = 1/N_m$ $\Gamma(s) = F(s)$  Parallel Combination of Resistance and Inverse Impedance

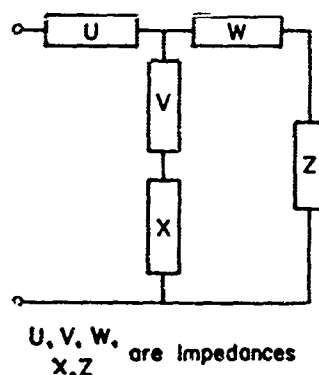


Figure 6a. The Terminated Perfectly Coupled and Shunt-Augmented T

Let

$$\bar{U} = u\phi(s) = v(n-1)\phi(s), \quad (77a)$$

$$\bar{V} = v\phi(s), \quad (77b)$$

$$\bar{W} = w\phi(s) = v\left(\frac{1}{n} - 1\right)\phi(s), \quad (77c)$$

$$\bar{X} = x\phi(s), \quad (77d)$$

$$\bar{Z} = z(s), \quad (77e)$$

then

$$\bar{Z}(s) = \bar{F}(s) \quad (78)$$

where $\bar{F}(s)$ is as in Eq. (61). With Eq. (76) of course Eq. (71) is also true.

Consider next Figure 6b. It consists of a Pi that is terminated by the branch Z. The Pi has the shunt branches U and W and a series branch that is composed of the branch elements V and X. The branches U, V, W make a coupled unit; thus we may say that X is an augmentation in the series branch of the Pi. Let all capital notations in Figure 6b be admittances. Let Eq. (75) be true. Then the driving-point admittance of the terminated Pi in Figure 6b is

$$\bar{F}(s) = \frac{X(U + W) + Z(U + V) + XZ}{(V + W) + (X + Z)}. \quad (79)$$

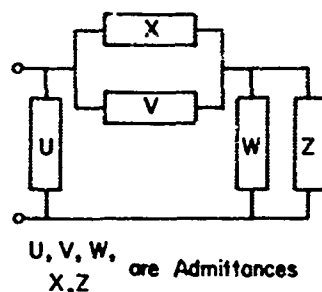


Figure 6b. The Terminated Perfectly Coupled and Series-Augmented Pi

Let the U, V, W, X, Z which are now admittances be determined as in Eqs. (77a, ..., e), then

$$I(s) = \bar{F}(s), \quad (80)$$

where $\bar{F}(s)$ is as in Eq. (61). The right side of Eq. (79) is the same as the right side in Eq. (76) since the structures in Figure 6a and Figure 6b are vice versa dual.

By Eq. (75) one branch element of the U, V, W is necessarily negative. We will see later that in dealing with biquadratic functions each branch element contains only one R, L , or C circuit element. But quite generally, no matter how the branches are composed, the negative branch contains RLC elements that are all negative. Thus the circuits represented in Figure 6a and Figure 6b cannot be realized from the set of positive R, L, C elements. We refer to them for this reason as model circuits. They can be realized as such by using negative circuit elements as far as necessary. They also can be expanded (as already mentioned in the Introduction) to circuits with exclusively positive elements, however this has to be paid by additional elements. The model circuits are canonical, the expanded circuits are not canonical. Realizations by using negative circuit elements and the expansions, however, are considered to be beyond the scope of the present paper.

5.3 RLC Ladders with Exclusively Positive Circuit Elements for the Realizations of Immittance Functions $F(s)$

All immittance functions of the types P_i, Q_i, P_i^{-1} , and Q_i^{-1} are realizable in canonical form as RC structures in one interpretation of the function and as RL structures in the other interpretation. More precisely, P_i and Q_k^{-1} ($k = \text{or } \neq i$) are RC (or RL) structures and P_i^{-1} and Q_k are RL (or RC) structures in the same interpretation of the function. In the other interpretation of the function, P_i and

Q_k^{-1} are RL (or RC) structures and P_i^{-1} and Q_k are RC (or RL) structures. Therefore realizations of the types Q_4^{-1} and P_4^{-1} appear when $F(s)$ and $F'(s)$ are both of the sub-categories (1). The realization is either (depending on the interpretation of $F(s)$) RC or RL structures. But when the decomposition implies the function $(Q_1 + P_3)$ or the function $(Q_1^{-1} + P_3^{-1})$, then the realization can only be a RLC network.

Realizations of the functions of the types P_i , Q_i , P_i^{-1} , and Q_i^{-1} are listed in Tables A-4, ..., A-11 in the appendix.

Table A-4 is provided for $i = 1$.

Table A-5 is provided for $i = 2$. We do not use this table here. This one and Table A-1 have been taken into the appendix more for the reason of a systematical completeness.

Tables A-6 and A-7 are provided for $i = 3$. In this case each realization for Table A-6 has a correspondent equivalent realization for Table A-7.

Tables A-8, ..., A-11 are provided for $i = 4$. In this case each realization for one table has a corresponding equivalent realization for each of the three other tables.

At the top of each of the Tables A-4, ..., A-11 we find six circuit realizations A_1 , B_1 , C_1 and A_1^* , B_1^* , C_1^* , and so forth. The asterisk indicates duality. Below the circuits in the respective circuit columns we find the impedance and the admittance interpretation of the type of function under consideration. Finally, the lower part of each table gives the formula for computing the circuit elements based on the function parameters.

Let us now get a merely qualitative picture of what kind of circuit realizations we get when we interpret $F(s) = Z(s)$, and when $F(s)$ is of the odd sub-categories (1) or (3).

Let $F(s)$ be of category (a1) and $F'(s)$ of category (b1). Then by the upper right part of Table 8a we are advised to link the impedance Q_4^{-1} in series with the resistance R_0 . This impedance has four equivalences, the circuits B_5^* , ..., B_8^* on Tables A-8, ..., A-11. Thus we obtain the circuits D_1 , ..., D_4 on Table 11. Each of these four circuits is equivalent with any other.

Let $F(s)$ be of category (a10) or (a3) but $F'(s)$ of category (b4). Then by the lower right part of Table 8a we are advised to link the inverse admittance $(Q_1 + P_3)$ in series with the resistance R_0 . We find the admittance Q_1 as circuit B_1^* on Table A-4 and the admittance P_3 as circuit C_3^* on Table A-6 and as circuit C_4^* on Table A-7. These admittances must be combined in parallel. Thus we obtain the equivalent circuits D_5 and D_6 on Table 11.

Let $F(s)$ be of category (b1) and $F'(s)$ be of category (a1). Consulting the upper right part of Table 9a we are advised to link an impedance P_4^{-1} in series with the resistance R_0 . We find four equivalent circuits with the impedance P_4^{-1}

as circuits C_5^* , ..., C_8^* on Tables A-8, ..., A-11. Thus we obtain the four equivalent circuits D_7 , ..., D_{10} on Table 11.

Let $F(s)$ be of category (b1) or (b3), but $F'(s)$ of category (a4). By the lower right part of Table 9a we are advised to link the inverse admittance $(Q_1^{-1} + P_3^{-1})$ in series with the resistance R_0 . The admittance Q_1^{-1} is found as circuit C_1 on Table A-4. The admittance P_3^{-1} has the two equivalent circuits B_3 on Table A-6 and B_4 on Table A-7. The admittances have to be linked in parallel. Thus we obtain the two equivalent circuits D_{11} and D_{12} in Table 11.

Let us now interpret $\Gamma(s) = F'(s)$ and assume that again $F(s)$ is of the odd sub-categories (1) or (3). Evidently, we have to obtain the dual circuits. It is up to the reader to check that the following statements are correct:

If $F(s)$ is of category (a1) and $F'(s)$ is of category (b1) we obtain the circuits D_1^* , ..., D_4^* on Table 12 which are all equivalent.

If $F(s)$ is of category (a1) or (a3) and $F'(s)$ is of category (b4), we obtain the two equivalent circuits D_5^* and D_6^* on Table 12.

If $F(s)$ is of category (b1) and $F'(s)$ of category (a1), we obtain the four equivalent circuits D_7^* , ..., D_{10}^* on Table 12.

If $F(s)$ is of category (b1) or (b3) but $F'(s)$ is of category (a4), we obtain the two equivalent circuits D_{11}^* and D_{12}^* on Table 12.

Assume now that we consider the interpreted function as the predominant one in the particular event that the function and its inverse are both of the sub-category (1).

Let $Z(s) = F(s)$ be of category (a1) and $\Gamma(s)$ of category (b1). Then $Z(s)$ can be realized by the four equivalent circuits D_1 , ..., D_4 on Table 11 as we have found. But it can also be realized by the four equivalent circuits D_7^* , ..., D_{10}^* on Table 12. Thus there is a total of eight circuits for the realization of $Z(s)$.

Let $Z(s) = F(s)$ be of the category (b1) and $\Gamma(s)$ of the category (a1). Then $Z(s)$ can be realized by the four equivalent circuits D_1^* , ..., D_4^* in Table 12 and also by the four equivalent circuits D_7 , ..., D_{10} in Table 11. Thus also in this event there is a total of eight circuits for the realization of $Z(s)$.

Note also that the circuits D_1 , ..., D_4 in Table 11 and the circuits D_7^* , ..., D_{10}^* in Table 12 are RC circuits, and the circuits D_7 , ..., D_{10} in Table 11 and the circuits D_1^* , ..., D_4^* in Table 12 are RL circuits.

When $Z(s) \equiv F(s)$ is of category (a1) or (a3), but $\Gamma(s)$ of category (b4), we obtain the circuits D_5 and D_6 ; when $Z(s) \equiv F(s)$ is of category (b1) or (b3) and $\Gamma(s)$ of category (a4), we obtain the circuits D_{11} and D_{12} . Both pairs of circuits are RLC structures and are found in Table 11.

When $\Gamma(s) \equiv F(s)$ is of category (a1) or (a3) and $Z(s)$ is of category (b4), we obtain the circuits D_5^* and D_6^* ; when $\Gamma(s) \equiv F(s)$ is of category (b1) or (b3) and $Z(s)$ is of category (a4), we obtain the circuits D_{11}^* and D_{12}^* . Both pairs of equivalent circuits are RLC structures and are found in Table 12.

Table 11. RLC Realizations with Exclusively Positive Circuit Elements (Duals of the Circuits in Table 12)

D_1 	D_2 	D_3 	D_4
D_5 	D_6 		
D_7 	D_8 	D_9 	D_{10}
D_{11} 	D_{12} 		

Table 12. RLC Realizations with Exclusively Positive Circuit Elements (Duals of the Circuits in Table 11)

D_1^* 	D_2^* 	D_3^* 	D_4^*
D_5^* 	D_6^* 		
D_7^* 	D_8^* 	D_9^* 	D_{10}^*
D_{11}^* 	D_{12}^* 		

Right now we are not yet in the position to give the other model circuits that exist besides the circuits D_5 , D_6 and D_{11} , D_{12} and circuits D_5^* , D_6^* and D_{11}^* , D_{12}^* . These model circuits will be developed in Section 5.4.

For convenient use in realization problems we summarize the instructions to obtain the circuits in Tables 11 and 12 and in the following Tables 13a, b. Table 13a is devised for the impedance interpretation $Z(s) \equiv F(s)$ and Table 13b for the admittance interpretation $\Gamma(s) \equiv F(s)$.

Table 13a. Realizations of the Impedance Interpretation $Z(s) \equiv F(s)$

$Z(s)$ of (a1) $\Gamma(s)$ of (b1)	Circuits D_1, \dots, D_4 $R_0 = 1,$ $\left. \begin{matrix} R_1, R_2, \\ C_1, C_2 \end{matrix} \right\}$ see Tables A-8, ..., A-11 for circuits B_5^*, \dots, B_8^*
$Z(s)$ of (a1) or (a3) $\Gamma(s)$ of (b4)	Circuits D_5 and D_6 $R_0 = 1$ C_1 see Table A-4 for circuit B_1^* $\left. \begin{matrix} R_1, R_2, \\ L_1 \end{matrix} \right\}$ see Tables A-6 and A-7 for circuits C_3^* and C_4^*
$Z(s)$ of (b1) $\Gamma(s)$ of (a1)	Circuits D_7, \dots, D_{10} $R_0 = D_0/N_0,$ $\left. \begin{matrix} R_1, R_2, \\ L_1, L_2 \end{matrix} \right\}$ see Tables A-8, ..., A-11 for circuits C_5^*, \dots, C_8^*
$Z(s)$ of (b1) or (b3) $\Gamma(s)$ of (a4)	Circuits D_{11} and D_{12} $R_0 = D_0/N_0,$ L_1 see Table A-4 for circuit C_1 $\left. \begin{matrix} R_1, R_2, \\ C_1 \end{matrix} \right\}$ see Tables A-6 and A-7 for circuits B_3 and B_4
Remarks: All circuit diagrams of the circuits D_1, \dots, D_{12} are on Table 11. L_1 in circuits D_5 and $D_6 = L_2$ in circuits C_3^* and C_4^* . C_1 in circuits D_{11} and $D_{12} = C_2$ in circuits B_3 and B_4 .	

Table 13b. Realizations of the Admittance Interpretation $\Gamma(s) \equiv F(s)$

$\Gamma(s)$ of (a1) $Z(s)$ of (b1)	Circuits D_1^*, \dots, D_4^* $R_0 = 1,$ $\left. \begin{matrix} R_1, R_2, \\ L_1, L_2 \end{matrix} \right\}$ see Tables A-8, ..., A-11 for circuits C_5, \dots, C_8
$\Gamma(s)$ of (a1) or (a3) $Z(s)$ of (b4)	Circuits D_5^* and D_6^* $R_0 = 1,$ L_1 see Table A-4 for circuit C_1 $\left. \begin{matrix} R_1, R_2, \\ C_1 \end{matrix} \right\}$ see Tables A-6 and A-7 for circuits B_3 and B_4
$\Gamma(s)$ of (b1) $Z(s)$ of (a1)	Circuits D_7^*, \dots, D_{10}^* $R_0 = N_0/D_0,$ $\left. \begin{matrix} R_1, R_2, \\ C_1, C_2 \end{matrix} \right\}$ see Tables A-8, ..., A-11 for circuits B_5, \dots, B_8
$\Gamma(s)$ of (b1) or (b3) $Z(s)$ of (a4)	Circuits D_{11}^* and D_{12}^* $R_0 = N_0/D_0,$ see Table A-4 for circuit B_1^* $\left. \begin{matrix} R_1, R_2, \\ L_1 \end{matrix} \right\}$ see Tables A-6 and A-7 for circuits C_3^* and C_4^*
Remarks: All circuit diagrams of the circuits D_1^*, \dots, D_{12}^* are on Table 12. C_1 in circuits D_5^* and $D_6^* = C_2$ in circuits B_3 and B_4 . L_1 in circuits D_{11}^* and $D_{12}^* = L_2$ in circuits C_3^* and C_4^* .	

5.4 Model Circuits in RLC Ladder Structure with One Negative Element for the Realizations of Immittance Fractions $F(s)$

We have shown in Section 5.2.3 that when an immittance function $F(s)$ is of one of the categories (a2), (b2), (a4), or (b4) and consequently in the decomposition Eq. (44c) the singular function $\bar{F}(s)$ is of the same category, and if $F(s)$ is interpreted as an impedance $Z(s) \equiv F(s)$, then the driving-point impedance of the circuit in Figure 6a is $\bar{Z}(s) \equiv \bar{F}(s)$. We also have shown that when $F(s)$ is

of one of the categories mentioned above, and when $F(s)$ is interpreted as an admittance $\Gamma(s) \equiv F(s)$, then the driving-point admittance of the circuit in Figure 6b is $\Gamma(s) \equiv \bar{F}(s)$. In the decomposition Eq. (44c), however, the singular function $\bar{F}(s)$ is multiplied by the factor $(1-N_m)$. But this means only that we have to multiply the U, V, W, X, Z by this factor to obtain the impedance $(1-N_m)\bar{Z}(s)$ and the admittance $(1-N_m)\bar{\Gamma}(s)$ respectively. We now give circuits realizing this impedance and admittance.

In deriving the function variations $\bar{F}_a, \bar{F}_b, \bar{F}_c$ and $\bar{F}_{ax}, \bar{F}_{bx}, \bar{F}_{cx}$ in Section 4.2.3 we used the normalized pr functions $s, 1/s$, and 1 . It follows from Tables A-2 and A-3 that these functions are of the types P_1 (or P_1^{-1}), Q_1 and Q_1^{-1} . It follows from Table A-4 that

<u>in impedance interpretation</u>	<u>in admittance interpretation</u>
$s \equiv$ inductance $L = 1$	$s \equiv$ capacitance $C = 1$
$\frac{1}{s} \equiv$ capacitance $C = 1$	$\frac{1}{s} \equiv$ inductance $L = 1$
$1 \equiv$ resistance $R = 1$	$1 \equiv$ resistance $R = 1$

By Table 5 and Eqs. (77a, ..., e) in impedance interpretation the U, V, W are

inductances	in \bar{F}_a	and \bar{F}_{ax} .
capacitances	in \bar{F}_b	and \bar{F}_{bx} .
resistances	in \bar{F}_c	and \bar{F}_{cx} .
X is a capacitance	in \bar{F}_a	and \bar{F}_{cx} .
an inductance	in \bar{F}_b	and \bar{F}_c .
a resistance	in \bar{F}_{ax}	and \bar{F}_{bx} .
Z is a resistance	in \bar{F}_a	and \bar{F}_b .
an inductance	in \bar{F}_{bx}	and \bar{F}_{cx} .
a capacitance	in \bar{F}_c	and \bar{F}_{ax} .

Also by Table 5 and Eqs. (77a, ..., e) in admittance interpretation the U, V, W are

capacitances	in \bar{F}_a	and \bar{F}_{ax} .
inductances	in \bar{F}_b	and \bar{F}_{bx} .
resistances	in \bar{F}_c	and \bar{F}_{cx} .
X is an inductance	in \bar{F}_a	and \bar{F}_{cx} .
a capacitance	in \bar{F}_b	and \bar{F}_c .
a resistance	in \bar{F}_{ax}	and \bar{F}_{bx} .
Z is a resistance	in \bar{F}_a	and \bar{F}_b .
a capacitance	in \bar{F}_{bx}	and \bar{F}_{cx} .
an inductance	in \bar{F}_c	and \bar{F}_{ax} .

These results are merely qualitative.

When we know the coefficients of $\bar{F}(s)$, then Table 6 gives us the constants u, v, w, x , and z . We only have to multiply these constants by the factor $(1-N_m)$ to get the branch impedances and branch admittances respectively; by the qualitative results above it is easy to obtain the sizes of the elements. Instructions for this are given by Table 14.

In the upper part of Table 14 we find the circuits F_1, \dots, F_6 with the driving-point impedance $(1-N_m)\bar{F}(s)$ and the dual circuits F_1^*, \dots, F_6^* with the driving-point admittance $(1-N_m)\bar{F}(s)$. The lower part of the table gives the RLC elements as the products or the inverse products of the constants u, v, w, x, z multiplied by the factor $(1-N_m)$.

All these circuits in Table 14 are model circuits since one of the circuit elements in the group of U, V, W is negative. The negative element is indicated by an encircled \ominus , resulting from the main-category (a) of $\bar{F}(s)$ and by an encircled \oplus , resulting from the main-category (b) of $\bar{F}(s)$.

We have shown in Section 4.2.3 that besides the decomposition Eq. (44c), the decomposition Eq. (59) is also possible. In this decomposition the functional component is $1/(1-N_m)' \bar{F}'(s)$. Thus the immittance function $(1-N_m)' \bar{F}'(s)$ has the opposite interpretation of $\bar{F}(s)$. The factor $(1-N_m)'$ is defined as the inverse of $(1-N_m)$. When we know $\bar{F}(s)$, we also know $\bar{F}'(s)$: we have only to interchange \bar{D}_0 with \bar{N}_0 and \bar{D}_1 with \bar{N}_1 . By Table 6 we obtain the u, v, w, x, z for $\bar{F}'(s)$ using the primed coefficients defined in Eqs. (60b, ..., e). Then we apply Table 14; but instead of the factor $(1-N_m)$ we have to use the factor $(1-N_m)' = 1/(1-N_m)$. To be correct, the impedance and admittance functions in this table also have to be read as primed notations in this application. Thus for any circuit F_1, \dots, F_6 derived from $\bar{F}(s)$ there is also an equivalent circuit F_1^*, \dots, F_6^* derived from $\bar{F}'(s)$ and vice versa. This is also verified by Tables 10a, b.

We are able to give the complete model circuits of immittance functions $F(s)$ that are of any of the categories (a2), (b2), (a4), (b4). As soon as $\bar{F}(s)$ is obtained as summarized in Table 7, it does not matter which one of these four categories is under consideration.

According to Tables 10a, b and Eqs. (44c) and (59)

$$F(s) = N_m + (1-N_m)\bar{F}(s) = N_m + \frac{1}{(1-N_m)' \bar{F}'(s)} \quad (81)$$

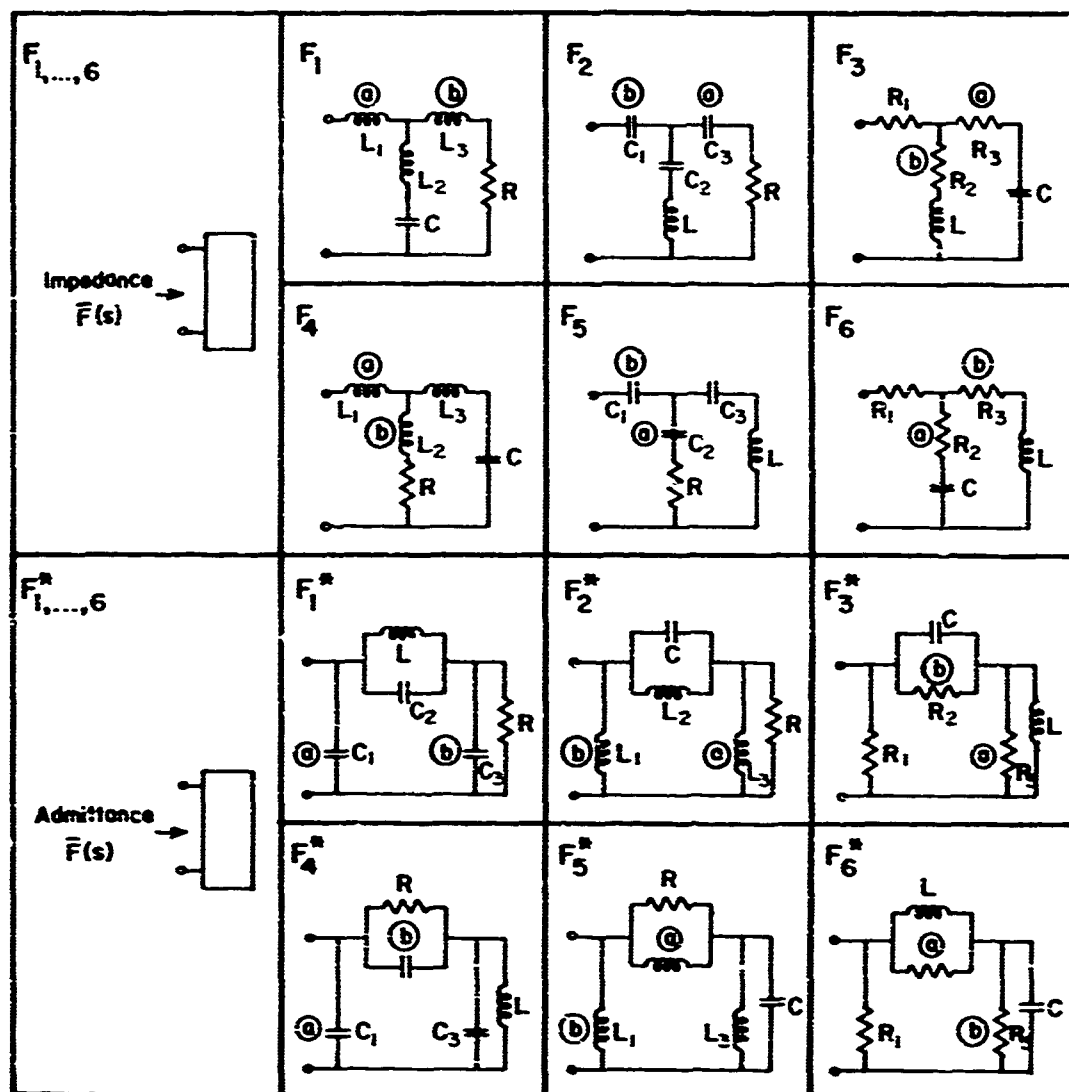
Thus for the

impedance interpretation $Z(s) \equiv F(s)$

$$Z(s) = R_C + \text{impedance } (1-N_m)\bar{F}(s), \quad (82a)$$

or

Table 14. Model Realizations of the Impedance $(1-N_m)\bar{F}(s)$ and the Admittance $(1-N_m)\bar{F}(s)$



F_1	F_4	F_2	F_5	F_3	F_6	F_1^*	F_4^*	F_2^*	F_5^*	F_3^*	F_6^*	Circuit
\bar{F}_a	\bar{F}_{ca}	\bar{F}_b	\bar{F}_{ba}	\bar{F}_c	\bar{F}_{ca}	$1/\bar{F}_a$	$1/\bar{F}_{ca}$	$1/\bar{F}_b$	$1/\bar{F}_{ba}$	$1/\bar{F}_c$	$1/\bar{F}_{ca}$	Impedance Function
$1/\bar{F}_a$	$1/\bar{F}_{ca}$	$1/\bar{F}_b$	$1/\bar{F}_{ba}$	$1/\bar{F}_c$	$1/\bar{F}_{ca}$	\bar{F}_a	\bar{F}_{ca}	\bar{F}_b	\bar{F}_{ba}	\bar{F}_c	\bar{F}_{ca}	Admittance Function
L_1		$1/C_1$		R_1		C_1		$1/L_1$		$1/R_1$		$x(1-N_m)$
L_2		$1/C_2$		R_2		C_2		$1/L_2$		$1/R_2$		
L_3		$1/C_3$		R_3		C_3		$1/L_3$		$1/R_3$		
$1/C$	R	L	R	L	$1/C$	$1/L$	$1/R$	C	$1/R$	C	$1/L$	x
R	$1/C$	R	L	$1/C$	L	$1/R$	$1/L$	$1/R$	C	$1/L$	C	z

$$Z(s) = R_0 + \frac{1}{\text{admittance } (1-N_m) \bar{F}'(s)} \quad (82b)$$

In Eqs. (82a, b)

$$R_0 = N_m. \quad (82c)$$

Talking in terms of circuitry

Eq. (82a) means

$$\text{resistance } R_0 \text{ in series with } \left\{ \begin{array}{l} \text{circuits } F_1, \dots, F_6 \\ \text{derived from} \\ \bar{F}(s) \text{ with factor } (1-N_m) \end{array} \right\} \quad (83a)$$

Eq. (82b) means

$$\text{resistance } R_0 \text{ in series with } \left\{ \begin{array}{l} \text{circuits } F_1^*, \dots, F_6^* \\ \text{derived from} \\ \bar{F}'(s) \text{ with factor } (1-N_m)' \end{array} \right\} \quad (83b)$$

The complete model circuits for (83a) are found as circuits M_1, \dots, M_6 in Table 15. The model circuits for (83b) are the circuits M_7^*, \dots, M_{12}^* in Table 16. All these twelve circuits are equivalent.

For the

admittance interpretation $I(s) \equiv F(s)$ of Eq. (81)

$$I(s) = \frac{1}{R_0} + \text{admittance } (1-N_m) \bar{F}(s), \quad (84a)$$

or

$$I(s) = \frac{1}{R_0} + \frac{1}{\text{impedance } (1-N_m) \bar{F}'(s)}. \quad (84b)$$

In Eqs. (84a, b)

$$R_0 = 1/N_m. \quad (84c)$$

Talking in terms of circuitry

Eq. (84a) means

$$\text{resistance } R_0 \text{ in parallel with } \left\{ \begin{array}{l} \text{circuits } F_1^*, \dots, F_6^* \\ \text{derived from} \\ \bar{F}(s) \text{ with factor } (1-N_m) \end{array} \right\} \quad (85a)$$

Table 15. Model Circuits M_1, \dots, M_6 and M_7^*, \dots, M_{12}^* (Duals of the Circuits in Table 16)

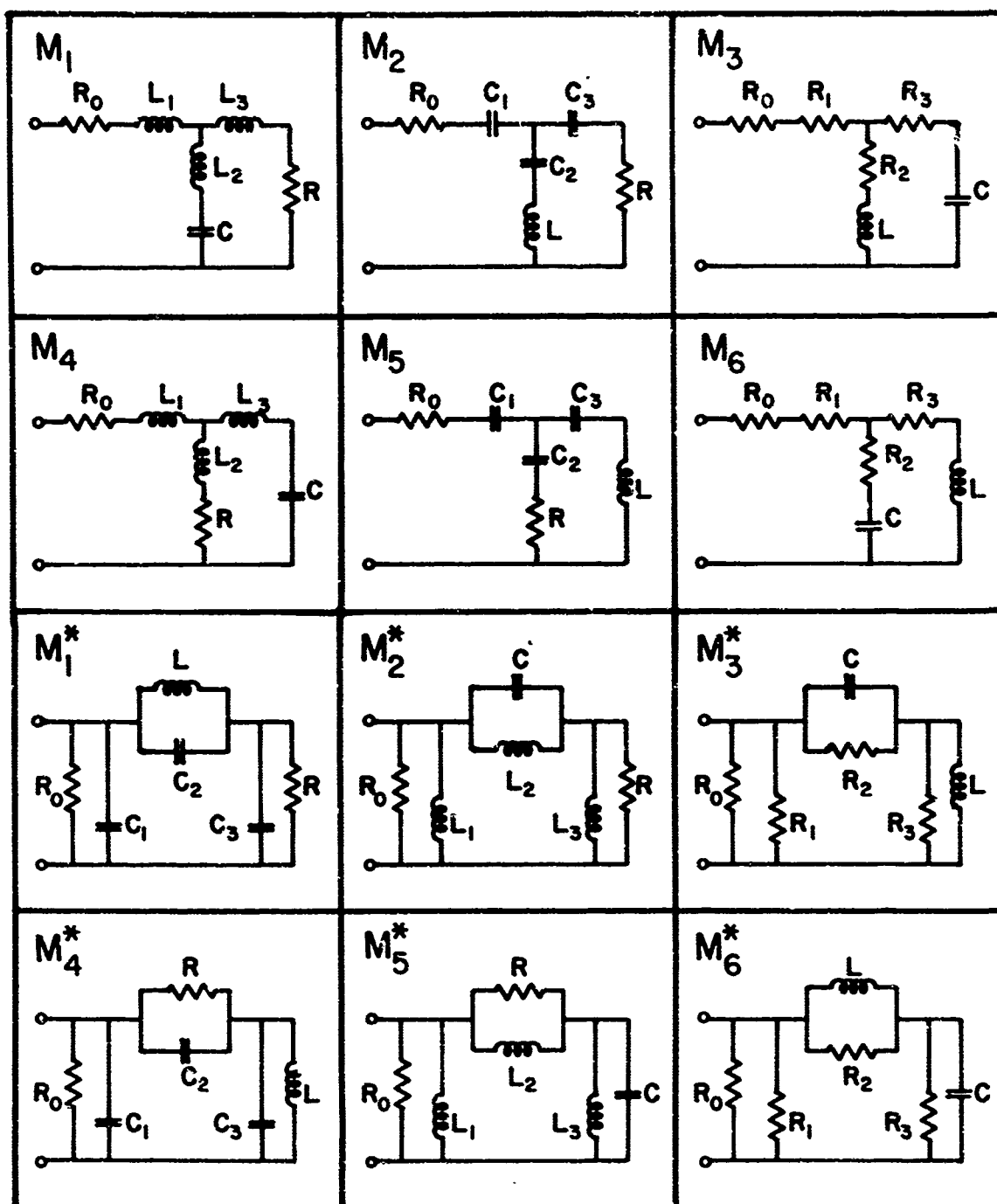
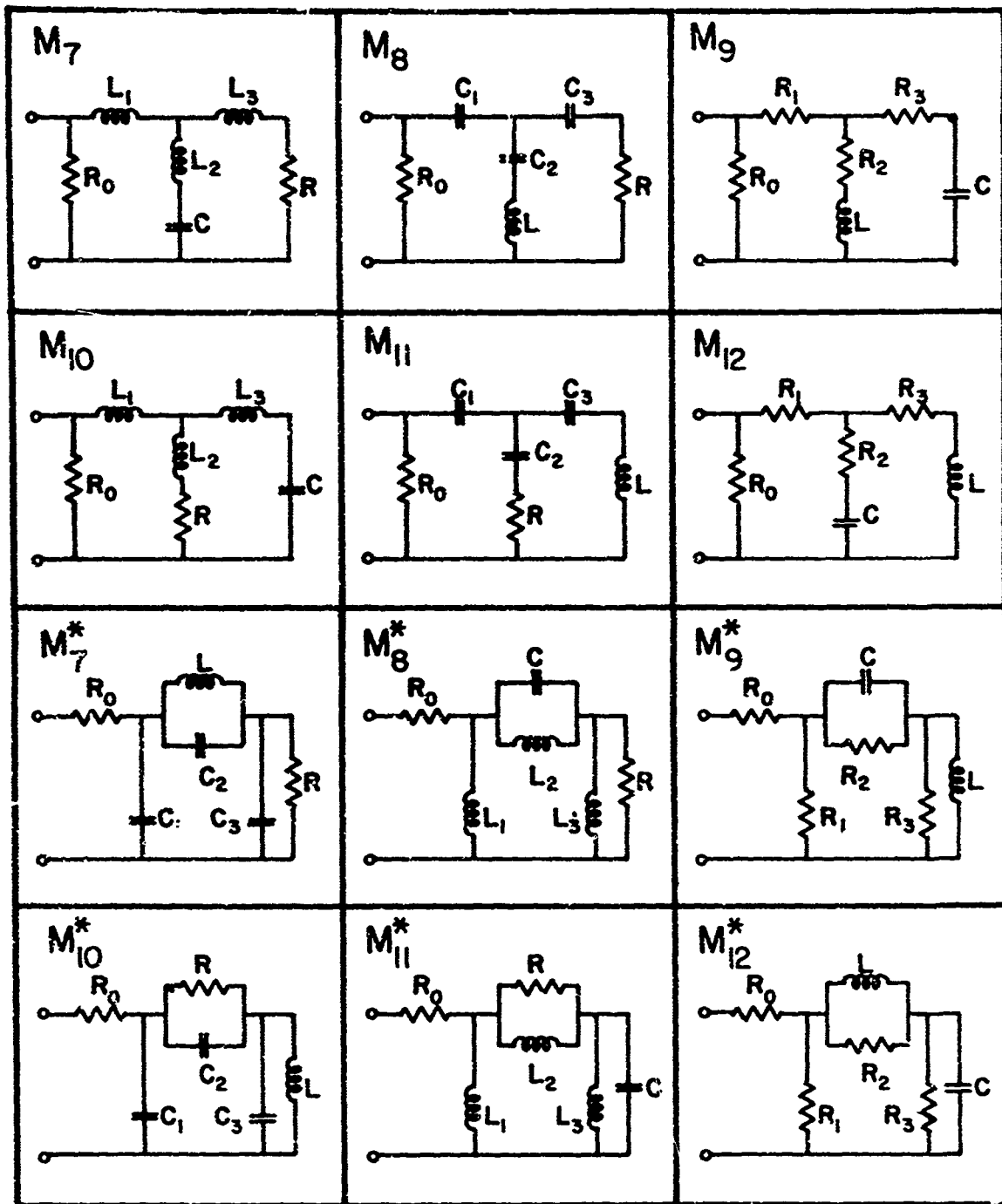


Table 16. Model Circuits M_1^*, \dots, M_6^* and M_7, \dots, M_{12} (Duals of the Circuits in Table 15)



Equation (84b) means

$$\text{resistance } R_0 \text{ in parallel with } \left\{ \begin{array}{l} \text{circuits } F_1, \dots, F_6 \\ \text{derived from} \\ \overline{F}'(s) \text{ with factor } (1-N_m)' \end{array} \right\} \quad (85b)$$

The complete model circuits for (85a) are found as circuits M_1^*, \dots, M_6^* in Table 15. The model circuits for (85b) are the circuits M_7, \dots, M_{12} in Table 16. All these twelve circuits are equivalent. The asterisks verify the duality between the impedance and the admittance interpretation.

For convenience in solving realization problems we summarize instructions for obtaining the circuits on Tables 15 and 16 in Tables 17a, b. Table 17a is devised for the impedance interpretation $Z(s) \equiv F(s)$, and Table 17b for the admittance interpretation $I(s) \equiv F(s)$. The function $F(s)$ is of sub-category (2) or (4).

When the immittance function $F(s)$ and its inverse $F'(s)$ are both of the even sub-categories (2) and/or (4), then we can obtain model circuits only in one or the other interpretation. Such a function has a total of 24 model circuits. They are all equivalent, but none of them is identical with any other. Such immittance functions are possible according to Tables 3a, b.

When either the immittance function $F(s)$ or its inverse $F'(s)$ are of the even sub-category (2) or (4), but the immittance function $F'(s)$ or its inverse $F(s)$ is of the odd sub-categories (1) or (3), then we obtain only 12 model circuits (either M_1, \dots, M_6 and M_7^*, \dots, M_{12}^* or M_1^*, \dots, M_6^* and M_7, \dots, M_{12}) and in addition to these equivalent RLC ladder structures with exclusively positive circuit elements. Such immittance functions are also possible according to Tables 3a, b. Thus we close a gap which we left open at the end of Section 5.3. The ladder structures can only be the circuits D_5, D_6 , or D_{11}, D_{12} in Table 11 or the circuits D_5^*, D_6^* , or D_{11}^*, D_{12}^* in Table 12. The reason is that the circuits D_1, \dots, D_4 and D_7, \dots, D_{12} in Table 11 and their duals in Table 12 postulate that $F(s)$ and $F'(s)$ are both of sub-category (1).

5.5 The Total of All Realization Possibilities of pr Biquadratic Immittance Functions in Canonical Ladder Form

Since we are able to adjust the immittance function $F(s)$ either to the impedance function or to the admittance function, we can decide that in the following investigation the function $F(s)$ shall always be of the main-category (a). By the interpretations $Z(s) \equiv F(s)$ and $I(s) \equiv F(s)$ we cover all possibilities. This decision, however, is in general not necessary.

Summarizing the contents of Tables 13a, b and 17a, b it is easy to establish Table 18. This table makes it convenient to get an immediate orientation on what kind of canonical circuitry one has to expect from an impedance function $Z(s)$ or

Table 17a. Realization of the Impedance Interpretation $Z(s) \equiv F(s)$

Category of $F(s)$: (a2), (b2), (a4), or (b4)	
(1)	<p>Use the coefficients $\bar{D}_0, \bar{D}_1, \bar{N}_0, \bar{N}_1$ and the factor $(1-N_m)$. Apply Table 6 to get the constants u, v, w, x, z. Apply Table 14 with these constants and the factor $(1-N_m)$ to get the circuits F_1, \dots, F_6. The first group are the realizing circuits M_1, \dots, M_6 in Table 15. Resistance $R_0 = N_m$, all other circuit elements as obtained for circuits F_1, \dots, F_6 by Table 14.</p>
(2)	<p>Use the coefficients $\bar{D}'_0, \bar{D}'_1, \bar{N}'_0, \bar{N}'_1$ and the factor $(1-N_m)' = 1/(1-N_m)$. Apply Table 6 assuming that all D and N are primed to get the constants u, v, w, x, z. Apply Table 14 with these constants and the factor $(1-N_m)'$ to get the circuits F_1^*, \dots, F_6^*. The second group are the realizing circuits M_7^*, \dots, M_{12}^* in Table 16. Resistance $R_0 = N_m$, all other circuit elements as obtained for circuits F_1^*, \dots, F_6^* by Table 14.</p>

Table 17b. Realization of the Admittance Interpretation $\Gamma(s) \equiv F(s)$

Category of $F(s)$: (a2), (b2), (a4), or (b4)	
(1)	<p>Use the coefficients $\bar{D}_0, \bar{D}_1, \bar{N}_0, \bar{N}_1$ and the factor $(1-N_m)$. Apply Table 6 to get the constants u, v, w, x, z. Apply Table 14 with these constants and the factor $(1-N_m)$ to get the circuits F_1^*, \dots, F_6^*. The first group are the realizing circuits M_1^*, \dots, M_6^* in Table 16. Resistance $R_0 = 1/N_m$, all other circuit elements as obtained for circuits F_1^*, \dots, F_6^* by Table 14.</p>
(2)	<p>Use the coefficients $\bar{D}'_0, \bar{D}'_1, \bar{N}'_0, \bar{N}'_1$ and the factor $(1-N_m)' = 1/(1-N_m)$. Apply Table 6 assuming that all D and N are primed to get the constants u, v, w, x, z. Apply Table 14 with these constants and the factor $(1-N_m)'$ to get the circuits F_1, \dots, F_6. The second group are the realizing circuits M_7, \dots, M_{12} in Table 15. Resistance $R_0 = 1/N_m$, all other circuit elements as obtained for circuits F_1, \dots, F_6 by Table 14.</p>

an admittance function $\Gamma(s)$, recognized by the numerical values of its coefficients. One has only to identify the categories of the function and of the inverse function by the test values T_0 , T_a , and T_1 for both functions and then one is able to find the realizations qualitatively in Table 18. In this respect the table is novel and a highly valuable source of information.

Table 18. All Realization Possibilities of pr Biquadratic Functions

$F(s)$	$F'(s)$	Impedance Interpretation $Z(s) \equiv F(s)$	Admittance Interpretation $I(s) \equiv F(s)$
(a1)	(b1)	D_1, \dots, D_4 D_7^*, \dots, D_{10}^*	D_1^*, \dots, D_4^* D_7, \dots, D_{10}
(a4)	(b1) or (b3)	D_{11}^*, D_{12}^* M_1, \dots, M_6 M_7^*, \dots, M_{12}^*	D_{11}, D_{12} M_1^*, \dots, M_6^* M_7, \dots, M_{12}
(a1) or (a3)	(b4)	D_5, D_6 M_1^*, \dots, M_6^* M_7, \dots, M_{12}	D_5^*, D_6^* M_1, \dots, M_6 M_7^*, \dots, M_{12}^*
(a2) or (a4)	(b2) or (b4)	M_1^*, \dots, M_6^* M_7, \dots, M_{12} M_1, \dots, M_6 M_7^*, \dots, M_{12}^*	M_1, \dots, M_6 M_7^*, \dots, M_{12}^* M_1^*, \dots, M_6^* M_7, \dots, M_{12}
Remarks: Note that $F(s)$ is adjusted and not the same in $Z(s)$ and $I(s)$. Find circuits D_1, \dots, D_{12} on Table 11 D_1^*, \dots, D_{12}^* on Table 12 M_1, \dots, M_6 on Table 15 M_1^*, \dots, M_6^* on Table 16 M_7, \dots, M_{12} on Table 16 M_7^*, \dots, M_{12}^* on Table 15			

Some remarks seem to be proper as to the row in the table where $F(s)$ is of category (a2) or (a4) and $F'(s)$ of category (b2) or (b4). In this case the circuits M_1^*, \dots, M_6^* and M_7, \dots, M_{12} result from the decomposition of $F(s)$ and the circuits M_1, \dots, M_6 and M_7^*, \dots, M_{12}^* result from a readjusted $F(s)$ where the D 's are interchanged with the N 's; by this all 24 circuits M_1, \dots, M_{12} and M_1^*, \dots, M_{12}^* are qualitatively obtained for the realization of $Z(s)$ and of $\Gamma(s)$.

As we recognize from Table 18 a pr biquadratic function of any constellation of categories offers quite a number of possibilities of realizations. In each interpretation we get a total of 8 RC or RL ladder realizations when the function and its inverse are of sub-category (1); there are two RLC ladder realizations with only positive circuit elements besides 12 model circuits when either the function or its inverse is of an even sub-category and the inverse or the direct function is of an even sub-category. There is a total of 24 model realizations, but no RLC realization with positive elements when the function and its inverse are of even sub-categories.

From the engineering point of view one may prefer one or several of the offered equivalent realizations and disregard the others. There is such a great variety to be judged that we do not intend to discuss eventual advantages of one circuit over others. This judgment is too dependent on the particular technical situation. But we want to point out that in particular among the circuits D_1, \dots, D_{12} , D_1^*, \dots, D_{12}^* and among the circuits M_1, \dots, M_{12} , M_1^*, \dots, M_{12}^* there are usually some whose circuit elements have acceptable dimensions and others that are not acceptable.

5.6 Discriminating Four Types of pr Functions

As regards their realizations Table 18 suggests the discrimination of four types of biquadratic and pr functions. When $F(s)$ and $F'(s)$ are both of the odd sub-category 1, then the realizations are ladder networks in both interpretations. We will refer to this type of function as "Type A/A." When $F(s)$ and $F'(s)$ are both of the even sub-categories 2 and/or 4, then the realizations are model circuits in both interpretations. We will refer to this second type of function as "Type B/B." When a function is of the odd sub-category 1 or 3 and its inverse is of the even sub-category 4, and when we realize the function, we will refer to it as "Type A/B."; if we realize the inverse function, we will refer to it as "Type B/A." Evidently, there is not much difference between the types A/B and B/A. The fraction notation only indicates that in the one interpretation the realization is a ladder network, in the other it is a model circuit. The type of discrimination thus is very general and not restricted to functions of the biquadratic rank.

6. PLANS FOR FURTHER STUDIES

We have decomposed and realized nine different numerical examples, each of them dealing with one typical constellation of categories of $F(s)$ and $F'(s)$. We felt that these results should be published separately. A separate publication is justified partly in order to avoid having the paper become too ambiguous, partly because we have also elaborated computation programs by which the realization of any pr biquadratic function can be performed rapidly and in a schematic way that necessitates only the use of a desk calculating machine. It is anticipated that this work will be published shortly after the appearance of the present paper.

Somewhat later we intend to publish a paper that deals with the further transformations of the model circuits into non-canonical RLC circuits without transformers but with positive elements.

References

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- Brune, Otto (1931) Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency, J. Math. and Phys. 10: 191-236.
- Haase, Kurt H. (1963a) Passive and Transformerless LC, RC, and RL One-Ports, AFCRL-63-506.
- Haase, Kurt H. (1963b) The Solution of Algebraic Equations by Root Finding Tables, Part I, Instructions, AFCRL-63-559 (I).
- Haase, Kurt H. (1964) The perfectly coupled and shunt-augmented T, a new and very general two-port model in passive network realizations, Proceedings of the Second Annual Allerton Conference on Circuit and System Theory, University of Illinois, Allerton House, Monticello, Illinois, September 28-30.
- Also published as: The Perfectly Coupled and Shunt-Augmented T Two-Port, AFCRL-65-215, April 1965.

Appendix A

The tables in Appendix A are reprints from
Haase, Kurt H. (1963) Passive and Transformerless LC, RC, and RL One-Ports,
AFCRL-63-506.

Table A1. Reactance Functions S_i and S_i^{-1} ($i = 1, 2, 3, 4$)

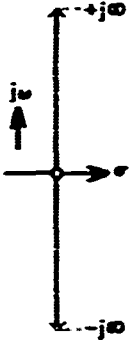
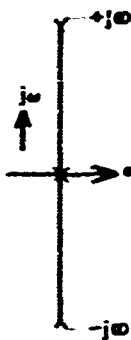
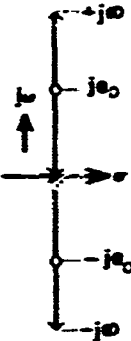
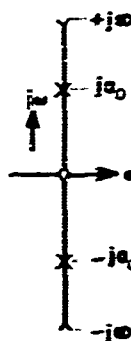
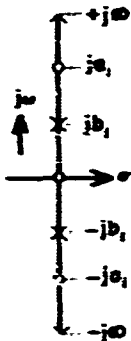
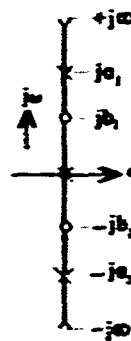
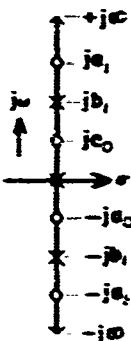
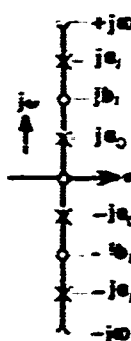
$S_1 = k \cdot s$ 	$S_1^{-1} = \frac{1}{k \cdot s}$ 
$S_2 = k \cdot \frac{s^2 + a_0^2}{s}$ 	$S_2^{-1} = \frac{1}{k} \cdot \frac{s}{s^2 + a_0^2}$ 
$S_3 = k \cdot s \cdot \frac{s^2 + a_1^2}{s^2 + b_1^2}$ 	$S_3^{-1} = \frac{1}{k \cdot s} \cdot \frac{s^2 + b_1^2}{s^2 + a_1^2}$ 
$S_4 = k \cdot \frac{s^2 + a_0^2}{s} \cdot \frac{s^2 + a_1^2}{s^2 + b_1^2}$ 	$S_4^{-1} = \frac{1}{k} \cdot \frac{s}{s^2 + a_0^2} \cdot \frac{s^2 + b_1^2}{s^2 + a_1^2}$ 

Table A2. Immittance Functions P_i and Q_i^{-1} ($i = 1, 2, 3, 4$)

$P_1 = k$	$Q_1^{-1} = \frac{1}{k \cdot s}$
$P_2 = k \cdot \frac{s + a_0^2}{s}$	$Q_2^{-1} = \frac{1}{k \cdot (s + a_0^2)}$
$P_3 = k \cdot \frac{s + a_1^2}{s + b_1^2}$	$Q_3^{-1} = \frac{1}{k \cdot s} \cdot \frac{s + b_1^2}{s + a_1^2}$
$P_4 = k \cdot \frac{s + a_0^2}{s} \cdot \frac{s + a_1^2}{s + b_1^2}$	$Q_4^{-1} = \frac{1}{k \cdot (s + a_0^2)} \cdot \frac{s + b_1^2}{s + a_1^2}$

Table A3. Immit'tance Functions P_i^{-1} and Q_i ($i = 1, 2, 3, 4$)

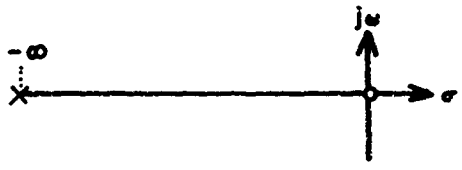

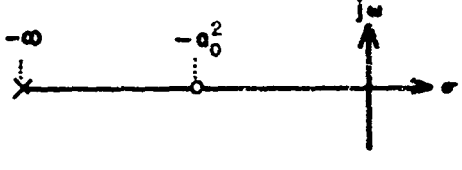
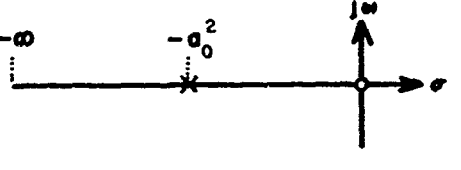
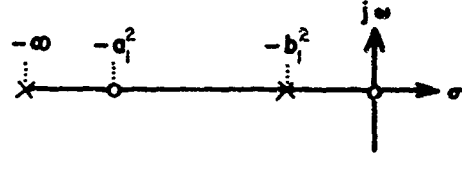
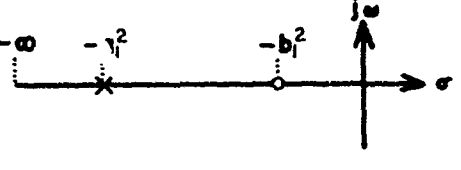
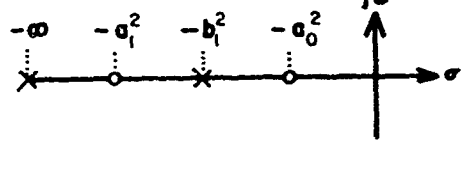
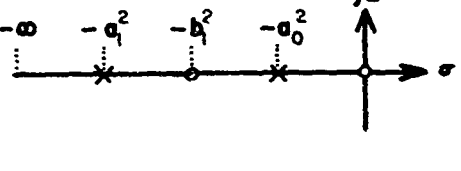
$Q_1 = k \cdot s$ 	$P_1^{-1} = \frac{1}{k}$ 
$Q_2 = k \cdot (s + a_0^2)$ 	$P_2^{-1} = \frac{1}{k} \cdot \frac{s}{s + a_0^2}$ 
$Q_3 = k \cdot s \cdot \frac{s + a_1^2}{s + b_1^2}$ 	$P_3^{-1} = \frac{1}{k} \cdot \frac{s + b_1^2}{s + a_1^2}$ 
$Q_4 = k \cdot (s + a_0^2) \cdot \frac{s + a_1^2}{s + b_1^2}$ 	$P_4^{-1} = \frac{1}{k} \cdot \frac{s}{s + a_0^2} \cdot \frac{s + b_1^2}{s + a_1^2}$ 

Table A4. Realizations $i = 1$

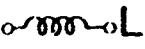

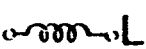
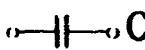
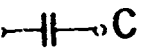

A_1			B_1			C_1
 L			 R			 L
A_1^*			B_1^*			C_1^*
 C			 C			 R
A_1	B_1	C_1	A_1^*	B_1^*	C_1^*	CIRCUIT
S_1	P_1	Q_1	S_1^{-1}	Q_1^{-1}	P_1^{-1}	IMPEDANCE FUNCTION
S_1^{-1}	P_1^{-1}	Q_1^{-1}	S_1	Q_1	P_1	ADMITTANCE FUNCTION
L	R	L	C	C	$\frac{1}{R}$	k

Table A5. Realizations $i = 2$

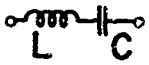





A_2 			B_2 			C_2 	
A_2^* 			B_2^* 			C_2^* 	
A_2	B_2	C_2	A_2^*	B_2^*	C_2^*	CIRCUIT	
S_2	P_2	Q_2	S_2^{-1}	Q_2^{-1}	P_2^{-1}	IMPEDANCE FUNCTION	
S_2^{-1}	P_2^{-1}	Q_2^{-1}	S_2	Q_2	P_2	ADMITTANCE FUNCTION	
L	R	L	C	C	$\frac{1}{R}$	k	
C	C	$\frac{1}{R}$	L	R	L	$\frac{1}{k \cdot a_0^2}$	
$\frac{1}{LC}$	$\frac{1}{RC}$	$\frac{R}{L}$	$\frac{1}{LC}$	$\frac{1}{RC}$	$\frac{R}{L}$	a_0^2	

Table A6. Realizations $i = 3$

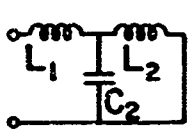
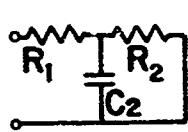
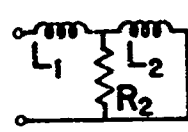
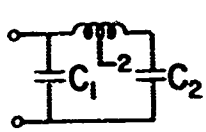
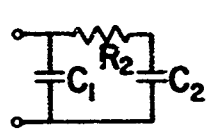
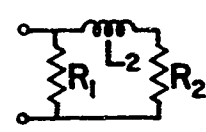
A₃ 			B₃ 			C₃ 		
A₃[*] 			B₃[*] 			C₃[*] 		
A₃	B₃	C₃	A₃[*]	B₃[*]	C₃[*]	CIRCUIT		
S₃	P₃	Q₃	S₃⁻¹	Q₃⁻¹	P₃⁻¹	IMPEDANCE FUNCTION		
S₃⁻¹	P₃⁻¹	Q₃⁻¹	S₃	Q₃	P₃	ADMITTANCE FUNCTION		
L₁	R₁	L₁	C₁	C₁	$\frac{1}{R_1}$	k		
L₂	R₂	L₂	C₂	C₂	$\frac{1}{R_2}$	$k \frac{a_1^2 - b_1^2}{b_1^2}$		
C₂	C₂	$\frac{1}{R_2}$	L₂	R₂	L₂	$\frac{1}{k} \frac{1}{a_1^2 - b_1^2}$		
$\frac{1}{L_2 C_2}$	$\frac{1}{R_2 C_2}$	$\frac{R_2}{L_2}$	$\frac{1}{L_2 C_2}$	$\frac{1}{R_2 C_2}$	$\frac{R_2}{L_2}$	b_1^2		

Table A7. Realizations $i = 3$

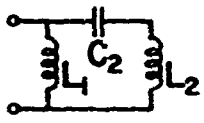
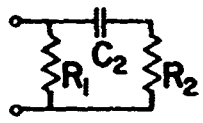
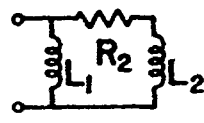
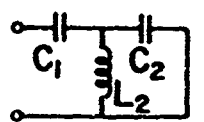
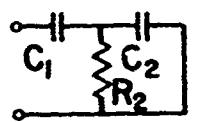
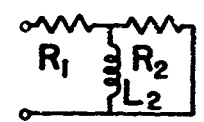
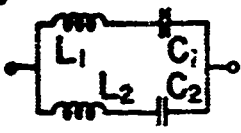
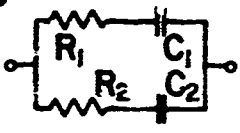
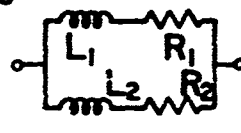
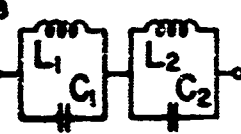
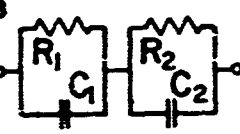
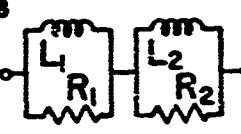
A₄ 			B₄ 			C₄ 		
A₄[*] 			B₄[*] 			C₄[*] 		
A₄	B₄	C₄	A₄[*]	B₄[*]	C₄[*]	CIRCUIT		
S₃	P₃	Q₃	S₃⁻¹	Q₃⁻¹	P₃⁻¹	IMPEDANCE FUNCTION		
S₃⁻¹	P₃⁻¹	Q₃⁻¹	S₃	Q₃	P₃	ADMITTANCE FUNCTION		
L₁	R₁	L₁	C₁	C₁	$\frac{1}{R_1}$	$k \frac{a_1^2}{b_1^2}$		
L₂	R₂	L₂	C₂	C₂	$\frac{1}{R_2}$	$k \frac{a_1^2}{a_1^2 - b_1^2}$		
C₂	C₂	$\frac{1}{R_2}$	L₂	R₂	L₂	$\frac{1}{k} \frac{a_1^2 - b_1^2}{a_1^4}$		
$\frac{1}{L_2 C_2}$	$\frac{1}{R_2 C_2}$	$\frac{R_2}{L_2}$	$\frac{1}{L_2 C_2}$	$\frac{1}{R_2 C_2}$	$\frac{R_2}{L_2}$	a_1^2		

Table A9. Realizations $i = 4$

A ₆	B ₆	C ₆	A ₆ [*]	B ₆ [*]	C ₆ [*]	CIRCUIT		
S ₄	P ₄	Q ₄	S ₄ ⁻¹	Q ₄ ⁻¹	P ₄ ⁻¹	IMPEDANCE FUNCTION		
S ₄ ⁻¹	P ₄ ⁻¹	Q ₄ ⁻¹	S ₄	Q ₄	P ₄	ADMITTANCE FUNCTION		
L ₁	R ₁	L ₁	C ₁	C ₁	$\frac{1}{R_1}$	$k \frac{a^4 + b_1^4}{b_1^4}$		
C ₁	C ₁	$\frac{1}{R_1}$	L ₁	R ₁	L ₁	$\frac{1}{k} \cdot \frac{b_1^2}{\rho a^4}$		
L ₂	R ₂	L ₂	C ₂	C ₂	$\frac{1}{R_2}$	$k \frac{a^4 + b_1^4}{a^4}$		
C ₂	C ₂	$\frac{1}{R_2}$	L ₂	R ₂	L ₂	$\frac{1}{k} \frac{b_1^2 a^4}{(a^4 + b_1^4)^2}$		
$\frac{1}{L_1 C_1}$	$\frac{1}{R_1 C_1}$	$\frac{R_1}{L_1}$	$\frac{1}{L_1 C_1}$	$\frac{1}{R_1 C_1}$	$\frac{R_1}{L_1}$	$\frac{\rho \cdot a^4 b_1^2}{a^4 + b_1^4}$		
$\frac{1}{L_2 C_2}$	$\frac{1}{R_2 C_2}$	$\frac{R_2}{L_2}$	$\frac{1}{L_2 C_2}$	$\frac{1}{R_2 C_2}$	$\frac{R_2}{L_2}$	$\frac{a^4 + b_1^4}{b_1^2}$		
$a^4 = b_1^2 (a_0^2 + a_1^2) - (b_1^4 + a_0^2 a_1^2)$						$\rho = \frac{a_0^2 a_1^2}{a^4}$		

Table A11. Realizations $i = 4$

								
								
A_0	B_0	C_0	A_0^*	B_0^*	C_0^*	CIRCUIT		
S_4	P_4	Q_4	S_4^{-1}	Q_4^{-1}	P_4^{-1}	IMPEDANCE FUNCTION		
S_4^{-1}	P_4^{-1}	Q_4^{-1}	S_4	Q_4	P_4	ADMITTANCE FUNCTION		
L_1	R_1	L_1	C_1	C_1	$\frac{1}{R_1}$	$k \cdot \frac{a_1^2 - a_0^2}{b_1^2 - a_0^2}$		
C_1	C_1	$\frac{1}{R_1}$	L_1	R_1	L_1	$\frac{1}{k} \cdot \frac{b_1^2 - a_0^2}{(a_1^2 - a_0^2) a_0^2}$		
L_2	R_2	L_2	C_2	C_2	$\frac{1}{R_2}$	$k \cdot \frac{a_1^2 - a_0^2}{a_1^2 - b_1^2}$		
C_2	C_2	$\frac{1}{R_2}$	L_2	R_2	L_2	$\frac{1}{k} \cdot \frac{a_1^2 - b_1^2}{(a_1^2 - a_0^2) a_1^2}$		
$\frac{1}{L_1 C_1}$	$\frac{1}{R_1 C_1}$	$\frac{R_1}{L_1}$	$\frac{1}{L_1 C_1}$	$\frac{1}{R_1 C_1}$	$\frac{R_1}{L_1}$	a_0^2		
$\frac{1}{L_2 C_2}$	$\frac{1}{R_2 C_2}$	$\frac{R_2}{L_2}$	$\frac{1}{L_2 C_2}$	$\frac{1}{R_2 C_2}$	$\frac{R_2}{L_2}$	a_1^2		

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	ROLE	WT	ROLE	WT	ROLE	WT
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